LECTURE 23: INTEGRATION ON MANIFOLDS

1. Top forms and orientability

¶ Top forms on manifolds.

Let M be a smooth manifold of dimension m. As we have known, $\Omega^k(M) = 0$ for $k > m$. Thus we will call any smooth m-form a *top form* on M. Now let $p \in M$ and (φ, U, V) a local chart near p. Then $dx^1 \wedge \cdots \wedge dx^m$ is a top form on U. Note that for any $q \in U$, $(dx^1 \wedge \cdots \wedge dx^m)_q \neq 0$ since at any $q \in U$,

$$(dx^1 \wedge \cdots \wedge dx^m)_q(\partial_1, \cdots, \partial_m) = \det(dx^i(\partial_j))_{1\le i,j\le m} = 1.$$

Moreover, since $\dim \Lambda^m T_pM = 1$, we see that for any top form ω on U and any $q \in U$, there is a real number λ_q such that

$$\omega_q = \lambda_q (dx^1 \wedge \cdots \wedge dx^m)_q.$$

Also by smoothness of ω, the coefficient λ, as a function on U, is smooth. So up to multiplication by functions, the "canonical top form" is the "only essential" top form in the chart. (However, this conclusion is not true globally on M: For two top forms $\omega, \eta \in \Omega^m(M)$, it may happen that $\omega_p = 0, \omega_q \neq 0$ while $\eta_p \neq 0, \eta_q = 0$, and thus there exists no function $f \in C^\infty(M)$ with $\omega = f\eta$ or $\eta = f\omega$.)

In particular, if we change coordinates from $(x_\alpha^1, \cdots, x_\alpha^m)$ to $(x_\beta^1, \cdots, x_\beta^m)$ on U, then we will get two top forms, $dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m$ and $dx_\beta^1 \wedge \cdots \wedge dx_\beta^m$. They should be related by a smooth function on U. It is not hard to find out this coordinate-change factor. We first prove

Lemma 1.1. *If $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is a diffeomorphism and $y = \varphi(x)$, then*

$$\varphi^*(dy^1 \wedge \cdots \wedge dy^m) = \det(d\varphi_x) dx^1 \wedge \cdots \wedge dx^m$$

Proof. If we denote $\varphi = (\varphi^1, \cdots, \varphi^m)$, then $\varphi^* y^i = y^i \circ \varphi = \varphi^i$. So

$$\varphi^*(dy^1 \wedge \cdots \wedge dy^m) = d\varphi^1 \wedge \cdots \wedge d\varphi^m.$$

But since

$$d\varphi^1 \wedge \cdots \wedge d\varphi^m(\partial_1^x, \cdots, \partial_m^x) = \det(d\varphi_x),$$

we conclude

$$d\varphi^1 \wedge \cdots \wedge d\varphi^m = \det(d\varphi_x) dx^1 \wedge \cdots \wedge dx^m.$$

□

Now let $(\varphi_\alpha, U, V_\alpha,)$ and $(\varphi_\beta, U, V_\beta)$ be two coordinate systems on U. Then the coordinate change map is $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$, which maps $\varphi_\alpha(x)$ to $y = \varphi_\beta(x)$. So we get

$$(\varphi_{\alpha\beta})^*(\varphi_\beta^{-1})^* dx_\beta^1 \wedge \cdots \wedge dx_\beta^m = \det(d\varphi_{\alpha\beta})(\varphi_\alpha^{-1})^* dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m.$$

Since $(\varphi_{\alpha\beta})^*(\varphi_\beta^{-1})^* = (\varphi_\beta^{-1} \circ \varphi_{\alpha\beta})^* = \varphi_\alpha^{-1}$, we arrive at

$$dx_\beta^1 \wedge \cdots dx_\beta^m = \det(d\varphi_{\alpha\beta}) dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m.$$

¶ **The need of orientability.**

Let M be a smooth manifold of dimension n, and let $\omega \in \Omega^m(M)$ be a smooth n-form. We want to define the integral $\int_M \omega$. For simplicity let's suppose ω is supported on a chart (φ, U, V) with coordinates $\{x^1, \cdots, x^m\}$. Then we can write

$$\omega = f(\varphi(x)) dx^1 \wedge \cdots \wedge dx^m,$$

where f is a smooth function on V. With the help of the Euclidean differential form $f(x)dx^1 \wedge \cdots \wedge dx^m$ on V, it is natural to define

$$\int_U \omega := \int_V f(x) dx^1 \cdots dx^m. \tag{1}$$

Then as usual one need to check that the integral in the right hand side is independent of the choice of coordinate charts.

So we let $(\varphi_\alpha, U, V_\alpha,)$ and $(\varphi_\beta, U, V_\beta)$ be two coordinate systems on U, with transition map $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \to V_\beta$ which maps $x_\alpha = \varphi_\alpha(x)$ to $x_\beta = \varphi_\beta(x)$. Then

$$\omega = f_\beta(x_\beta) dx_\beta^1 \wedge \cdots \wedge dx_\beta^m = f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(d\varphi_{\alpha\beta}) dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m$$

So for the definition to be well-defined, we need

$$\int_{V_\beta} f_\beta(x_\beta) dx_\beta^1 \cdots dx_\beta^m = \int_{V_\alpha} f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(d\varphi_{\alpha\beta}(x_\alpha)) dx_\alpha^1 \cdots dx_\alpha^m.$$

Unfortunately this is not always true: In calculus we learned that for the integrals of multi-variable functions

$$\int f(x) dx^1 \cdots dx^m,$$

if $\varphi : V_1 \to V_2$ is a diffeomorphism, then we have the *change of variable formula*: $(y = \varphi(x))$

$$\int_{V_2} f(x)\ dy^1 \cdots dy^m = \int_{V_1} f(\varphi(x)) |\det(d\varphi)(x)|\ dx^1 \cdots dx^m. \tag{2}$$

So we only have

$$\int_{V_\beta} f_\beta(x_\beta) dx_\beta^1 \cdots dx_\beta^m = \int_{V_\alpha} f_\beta(\varphi_{\alpha\beta}(x_\alpha))\, |\det(d\varphi_{\alpha\beta}(x_\alpha))|\, dx_\alpha^1 \cdots dx_\alpha^m.$$

In other words, for the definition to be independent of the choice of charts, we need to assume

$$\det(d\varphi_{\alpha\beta}) > 0$$

for all charts. In fact, we have seen this condition in PSet 2-1-3:

Definition 1.2. Let M be a smooth manifold of dimension n.

(1) Two charts $(\varphi_\alpha, U_\alpha, V_\alpha)$ and $(\varphi_\beta, U_\beta, V_\beta)$ are *orientation compatible* if the transition map $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfies

$$\det(d\varphi_{\alpha\beta})_p > 0$$

for all $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$.

(2) An *orientation* of M is an atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha, V_\alpha) \mid \alpha \in \Lambda\}$ whose charts are pairwise orientation compatible.
(3) We say M is *orientable* if it has an orientation.

Remark. Let U be a chart with coordinates $\{x^1, \cdots, x^m\}$. We use the notation $-U$ to represent the same coordinate chart U but with "twisted" coordinates $\{-x^1, x^2, \cdots, x^m\}$. Then $-U$ and U are not orientation compatible. Let $\widetilde{U}$ be any other coordinate chart such that $\widetilde{U} \cap U \neq \emptyset$ is *connected*. Then either

- $\widetilde{U}$ and U are orientation compatible,

or

- $\widetilde{U}$ and $-U$ are orientation compatible.

As a consequence, we immediately see

Corollary 1.3. *If M is connected and orientable, then M admits exactly two* different *orientations.*

Example. For the real projective space $\mathbb{RP}^n$, we have constructed an atlas consisting of $n+1$ charts. We have seen that $\mathbb{RP}^n$ is orientable for odd n. It turns out that $\mathbb{RP}^n$ is not orientable for even n.

2. Integrations on smooth manifolds

¶ Integrations of top forms on smooth manifolds.

Now assume M is a smooth orientable m-manifold and fix an orientation $\mathcal{A}$ on M. Let ω be any smooth m-form on M. To define $\int_M \omega$, we first assume that ω is supported in a coordinate chart (φ, U, V) which is orientation compatible with $\mathcal{A}$. In this case there is a function f supported in U such that

$$\omega = f(\varphi(x))dx^1 \wedge \cdots \wedge dx^m.$$

In this case we simply define

$$\int_U \omega := \int_V f(x)dx^1 \cdots dx^m, \tag{3}$$

where the right hand side is the Lebesgue integral on $V \subset \mathbb{R}^m$. We will assume f is integrable. In fact, in what follows the function f involved are compactly supported.

To integrate a general m-form $\omega \in \Omega^m(M)$, we take a locally finite cover $\{U_\alpha\}$ of M that are compatible with the orientation $\mathcal{A}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Now since each ρ_α is supported in U_α, each $\rho_\alpha\omega$ is supported U_α also. We define

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega. \tag{4}$$

We say that ω is *integrable* if the right hand side converges absolutely for any such cover and any such P.O.U. This is true, for example, if ω is compactly supported.

One need to check that the definition (4) above is independent of the choices of orientation-compatible coordinate charts, and is independent of the choices of partition of unity.

Theorem 2.1. *Suppose ω is compactly supported, or more generally, ω is integrable. The expression (4) is independent of the choices of $\{U_\alpha\}$ and the choices of $\{\rho_\alpha\}$.*

Proof. We first show that (3) is well-defined. The argument is essentially the same as in the Euclidian case: if ω is supported in U, and if $\{x^i_\alpha\}$ and $\{x^i_\beta\}$ are two orientation-compatible coordinate systems on U, so that

$$\omega = f_\alpha dx^1_\alpha \wedge \cdots \wedge dx^m_\alpha = f_\beta dx^1_\beta \wedge \cdots \wedge dx^m_\beta,$$

then we want to prove

$$\int_{V_\alpha} f_\alpha dx^1_\alpha \cdots dx^m_\alpha = \int_{V_\beta} f_\beta dx^1_\beta \cdots dx^m_\beta.$$

This is true, because

$$dx^1_\beta \wedge \cdots \wedge dx^m_\beta = \det(d\varphi_{\alpha\beta}) dx^1_\alpha \wedge \cdots \wedge dx^m_\alpha$$

implies $f_\alpha = \det(d\varphi_{\alpha\beta}) f_\beta$. Since $\det(d\varphi_{\alpha\beta}) > 0$, the conclusion follows from the change of variable formula in $\mathbb{R}^n$.

To prove (4) is well-defined, we suppose $\{U_\alpha\}$ and $\{U_\beta\}$ are two locally finite cover of M consisting of orientation-compatible charts, and $\{\rho_\alpha\}$ and $\{\rho_\beta\}$ are partitions of unity subordinate to $\{U_\alpha\}$ and $\{U_\beta\}$ respectively. Then $\{U_\alpha \cap U_\beta\}$ is a new locally finite cover of M, and $\{\rho_\alpha \rho_\beta\}$ is a partition of unity subordinate to this new cover. It is enough to prove

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha \cap U_\beta} \rho_\alpha \rho_\beta \omega.$$

This is true because for each fixed α,

$$\int_{U_\alpha} \rho_\alpha \omega = \int_{U_\alpha} (\sum_\beta \rho_\beta) \rho_\alpha \omega = \sum_\beta \int_{U_\alpha \cap U_\beta} \rho_\beta \rho_\alpha \omega.$$

□

¶ Change of variable formula.

Finally we extend the change of variable formula form $\mathbb{R}^m$ to manifolds.

Definition 2.2. Let M, N be orientable smooth n-manifolds, with orientations $\mathcal{A}$ and $\mathcal{B}$ respectively. A diffeomorphism $\varphi : M \to N$ is said to be *orientation-preserving* if for each $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, the chart $(\psi_\beta \circ \varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is orientation compatible with $\mathcal{A}$.

Suppose M, N are connected. It is easy to see that a diffeomorphism $\varphi : M \to N$ is orientation-preserving if and only if there exists <u>one</u> chart $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, such that the chart $(\psi_\beta \circ \varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is orientation compatible with $\mathcal{A}$. Similarly if there exists one chart $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, such that the chart $(\psi_\beta \circ \varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is incompatible with $\mathcal{A}$, then for every chart $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, the chart $(\psi_\beta \circ$

$\varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is incompatible with $\mathcal{A}$. In this case we say φ is *orientation-reverting.*

Now we state:

Theorem 2.3 (The change of variable formula.)**.** *Suppose M, N are n-dimensional orientable smooth manifolds, and $\varphi : M \to N$ is a diffeomorphism.*

(1) *If φ is an orientation-preserving, then*

$$\int_M f^*\omega = \int_N \omega.$$

(2) *If φ is an orientation-reverting, then*

$$\int_M f^*\omega = -\int_N \omega.$$

Proof. It is enough to prove this in local charts, in which case this is merely the change of variable formula in $\mathbb{R}^m$. □

Remark. If ω is a compactly supported k-form on M, where $k < m = \dim M$, then one cannot integrate ω over M. However, for any k-dimensional orientable submanifold $X \subset M$, one can define $\int_X \omega$ by setting it to be $\int_X \iota^*\omega$, where $\iota : X \hookrightarrow M$ is the inclusion map. By this way we get a "pairing" between k-forms on M and k-dimensional orientable submanifolds in M.

Remark. If M is not orientable, one cannot define integrals of differential forms as above. However, we can still integrate via *densities.* (c.f. J. Lee, Introduction to smooth manifolds, page 427-432.)

¶ Volume form and volume measure.

Next we show that orientability can be characterized via the existence of specific top forms:

Theorem 2.4. *An m-dimensional smooth manifold M is orientable if and only if M admits a nowhere vanishing smooth m-form μ.*

Proof. First let μ be a nowhere vanishing smooth m-form on M. Then on each local chart $(U, x^1, \cdots, x^m)$ (where U is always chosen to be connected), there is a smooth function $f \neq 0$ so that $\mu = f dx^1 \wedge \cdots \wedge dx^m$. It follows that

$$\mu(\partial_1, \cdots, \partial_m) = f \neq 0.$$

We can always take such a chart near each point so that $f > 0$, otherwise we can replace x^1 by $-x^1$. Now suppose $(U_\alpha, x^1_\alpha, \cdots, x^m_\alpha)$ and $(U_\beta, x^1_\beta, \cdots, x^m_\beta)$ be two such charts, so that on the intersection $U_\alpha \cap U_\beta$ one has

$$f dx^1_\alpha \wedge \cdots \wedge dx^m_\alpha = \mu = g dx^1_\beta \wedge \cdots \wedge dx^m_\beta = g \det(d\varphi_{\alpha\beta}) dx^1_\alpha \wedge \cdots \wedge dx^m_\alpha.$$

where $f, g > 0$. It follows that $\det(d\varphi_{\alpha\beta}) > 0$. So the atlas constructed by this way is an orientation.

Conversely, suppose $\mathcal{A}$ is an orientation. For each local chart U_α in $\mathcal{A}$, we let

$$\mu_\alpha = dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m.$$

Pick a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. We claim that

$$\mu := \sum_\alpha \rho_\alpha \mu_\alpha$$

is a nowhere vanishing smooth m-form on M. In fact, for each $p \in M$, there is a neighborhood U of p so that the sum $\sum_\alpha \rho_\alpha \mu_\alpha$ is a finite sum $\sum_{i=1}^k \rho_i \mu_i$. It follows that near p,

$$\mu(\partial_1^1, \cdots, \partial_m^1) = \sum_{i=1}^k (\det d\varphi_{1i})\rho_i > 0.$$

So $\mu \neq 0$ near p. $\square$

Definition 2.5. A nowhere vanishing smooth m-form μ on an m-dimensional smooth manifold M is called a *volume form*.

Remark. If M is orientable, and μ is a volume form, then the two orientations of M are represented by μ and $-\mu$ respectively. We denote the two orientations by $[\mu]$ and $[-\mu]$.

Remark. Let μ be a volume form on M, and the orientation on M is chosen to be $[\mu]$. Then we can define a linear functional

$$I : C_c(M) \to \mathbb{R}, \quad f \mapsto \int_M f\mu.$$

(Here, $C_c(M)$ represents the space of continuous functions with compact supports on M. Obviously the integrals above still make sense even if f is not smooth.) Since the orientation on M is chosen to be $[\mu]$, we see the functional I is positive, i.e. $I(f) \geq 0$ for $f \geq 0$. Since any manifold is both locally compact and σ-compact, the Riesz representation theorem implies that there exists a unique Radon measure (=a locally finite, regular measure defined on all Borel sets) m_μ such that

$$I(f) = \int_M f dm_\mu.$$

Using the measure $d\mu$, one can define function spaces like $L^p(M, \mu)$.

Remark. In particular, on any Lie group, one can define conceptions like left-invariant differential forms. Since any Lie group is orientable (Exercise), there exists left-invariant volume form on any Lie group G. The measures associated to left-invariant volume form on Lie groups are called *Haar measures*.