1. The De Rham cohomology

¶ Closed and exact forms.

We start with the following definition:

**Definition 1.1.** Let $M$ be a smooth manifold, and $\omega \in \Omega^k(M)$ is a $k$-form.

1. We say $\omega$ is *closed* if $d\omega = 0$.
2. We say $\omega$ is *exact* if there exists a $(k-1)$-form $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$.

Denote the set of closed $k$-forms by $Z^k(M)$, and the set of exact $k$-forms by $B^k(M)$:

- $Z^k(M) = \{ \text{the set of closed } k \text{-forms} \} = \text{ker}(d : \Omega^k(M) \to \Omega^{k+1}(M))$,
- $B^k(M) = \{ \text{the set of exact } k \text{-forms} \} = \text{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$.

As we have seen, the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a linear map so that for any $k$ and any $\omega \in \Omega^k(M)$,

$$d^2 \omega = d(d\omega) = 0.$$

So we have the following inclusion relation (as vector spaces and as additive groups)

$$B^k(M) \subset Z^k(M) \subset \Omega^k(M).$$

**Remark.** Suppose $\dim M = m$. Then by definition we have

- For $k > m$: $B^k(M) = Z^k(M) = \{ 0 \}$.
- For $k = 0$: $B^0(M) = \{ 0 \}$, and

$$Z^0(M) = \{ f \in C^\infty(M) \mid df = 0 \} \simeq \mathbb{R}^K.$$

where $K$ is the number of connected components of $M$.
- For $k = m$: $Z^m(M) = \Omega^m(M)$.

**Example.** Consider $M = \mathbb{R}$. We have

$$B^0(\mathbb{R}) = \{ 0 \}, \quad Z^0(\mathbb{R}) \simeq \mathbb{R} \quad \text{and} \quad \Omega^0(\mathbb{R}) = C^\infty(\mathbb{R}).$$

For any 1-form $g(t)dt$ on $\mathbb{R}$, we have

$$\omega = g(t)dt \iff \omega = dG, \text{ where } G(t) = \int_0^t g(\tau)d\tau.$$

It follows that $\Omega^1(\mathbb{R}) = B^1(\mathbb{R}) = Z^1(\mathbb{R})$. 

The De Rham cohomology groups.

Since $d^2 = 0$, we get the following de Rham cochain complex

\[ 0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0. \]

(Such a sequence of vector spaces [could be an infinite sequence] connected by a sequence of linear maps, such that the composition of any two consecutive maps is the zero map, is called a chain complex or a cochain complex, depending on the “direction” of the maps.)

**Definition 1.2.** The quotient group (vector space)

\[ H^k_{dR}(M) := Z^k(M)/B^k(M) \]

is called the $k$th de Rham cohomology group of $M$.

**Example.** For $M = \mathbb{R}$, we easily see $H^0_{dR}(\mathbb{R}) \simeq \mathbb{R}$ and $H^k_{dR}(\mathbb{R}) = \{0\}$ for $k \geq 1$.

Given any $\omega \in Z^k(M)$, we will denote by $[\omega]$ the corresponding cohomology class.

**Remark.** Suppose $\dim M = m$. According to the above remark, we get

\[ H^k_{dR}(M) = \{0\}, \quad \forall k > m \]

and (we still denote by $K$ the number of connected components of $M$)

\[ H^0_{dR}(M) \simeq \mathbb{R}^K. \]

By definition, $H^k_{dR}(M)$ is a vector space. We will see that for many smooth manifolds (including all compact manifolds),

\[ \dim H^k_{dR}(M) < \infty \]

for all $k$. On the other hand, we have $\dim H^0_{dR}(\mathbb{Z}) = \infty$. (As another example: it is not hard to prove $\dim H^1_{dR}(\mathbb{R}^2 \setminus \mathbb{Z}^2) = +\infty$.)

**Definition 1.3.** In the case $\dim H^k_{dR}(M) < \infty$ for each $k$, we will call the number

\[ b_k(M) = \dim H^k_{dR}(M) \]

the $k$th Betti number of $M$, and the number

\[ \chi(M) = \sum_{k=0}^m (-1)^k b_k(M) \]

the Euler characteristic of $M$.

The De Rham cohomology groups of $S^1$.

Consider $M = S^1$. As we have seen,

\[ H^0_{dR}(S^1) \simeq \mathbb{R} \quad \text{and} \quad H^k_{dR}(S^1) = 0 \quad \text{for} \quad k \geq 2. \]

To calculate $H^1_{dR}(S^1)$, we argue as in the previous example. Note that on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, the “angle” variable $\theta$ is not a globally defined smooth function on $S^1$, but the translation invariance of $d$ on $\mathbb{R}$ implies that the differential form $d\theta$ is a globally defined 1-form on $S^1$. (As a consequence, the 1-form $d\theta$ is a closed 1-form on $S^1$, but
is NOT an exact 1-form on $S^1$. However, it is an exact 1-form on any proper subset of $S^1$.) So we can write

$$Z^1(S^1) = \Omega^1(S^1) = \{f d\theta \mid f \in C^\infty(S^1)\}$$

$$\simeq \{f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi\}.$$

On the other hand, by the fundamental theorem of calculus,

$$\omega \text{ is an exact 1-form} \iff \omega = df, \text{ where } f \text{ is periodic with period } 2\pi$$

$$\iff \omega = g(\theta)d\theta, \text{ where } g \text{ is periodic with period } 2\pi \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0.$$

So we conclude

$$H^1_{dR}(S^1) \simeq \{f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi\} \setminus \{g \in C^\infty(\mathbb{R}) \mid g \text{ is periodic with period } 2\pi, \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0\}.$$

This implies that

$$H^1_{dR}(S^1) \simeq \mathbb{R},$$

since the linear map

$$\varphi : H^1_{dR}(S^1) \rightarrow \mathbb{R}, \quad [f] \mapsto \int_0^{2\pi} f(\theta)d\theta.$$

is an linear isomorphism:

- $\varphi$ is well-defined:

  $$[f_1] = [f] \implies f_1 - f \in B^1(S^1) \implies \int_0^{2\pi} f_1(\theta)d\theta = \int_0^{2\pi} f(\theta)d\theta.$$

- $\varphi$ is injective:

  $$[f_1] \neq [f] \implies f_1 - f \not\in B^1(S^1) \implies \int_0^{2\pi} f_1(\theta)d\theta \neq \int_0^{2\pi} f(\theta)d\theta.$$

- $\varphi$ is surjective: for any $c \in \mathbb{R}$,

  $$f(\theta) := c \in Z^1(S^1) \implies \varphi([f]) = \int_0^{2\pi} f(\theta)d\theta = 2\pi c.$$

### Operations on cohomology classes.

One can extend the wedge product and pull-back operations on differential forms to operations on cohomology classes.

First let $\omega \in Z^k(M)$ and $\eta \in Z^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = 0,$$

i.e. $\omega \wedge \eta \in Z^{k+l}(M)$. Moreover, for any $\xi_1 \in \Omega^{k-1}(M)$ and $\xi_2 \in \Omega^{l-1}(M)$,

$$\omega + d\xi_1 \wedge (\eta + d\xi_2) = \omega \wedge \eta + d \left[(-1)^k \omega \wedge \xi_2 + (-1)^{k-1} \xi_1 \wedge \eta + (-1)^{k-1} \xi_1 \wedge d\xi_2\right].$$

In other words, $[\omega \wedge \eta]$ is independent of the choice of $\omega$ and $\eta$ in $[\omega]$ and $[\eta]$. So we can define
Definition 1.4. The cup product between $[\omega] \in H^k_{dR}(M)$ and $[\eta] \in H^l_{dR}(M)$ is

$[\omega] \cup [\eta] := [\omega \wedge \eta] \in H^{k+l}_{dR}(M)$.

Similarly suppose $\varphi : M \to N$ is smooth. Then the fact $d\varphi^* = \varphi^* d$ implies

$\varphi^*(Z^k(N)) \subset Z^k(M)$ and $\varphi^*(B^k(N)) \subset B^k(M)$.

It follows that $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ descends to a pull-back $\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$:

$\varphi^*([\omega]) := [\varphi^* \omega]$.

Obviously $\varphi^*$ is a group homomorphism. It is easy to check

- $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
- $\text{Id}^* = \text{Id}$.

As an immediate consequence, we see that the de Rham cohomology groups are invariant under diffeomorphisms:

Corollary 1.5 (Diffeomorphism Invariance). If $\varphi : M \to N$ is a diffeomorphism, then

$\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$

is a linear isomorphism for all $k$. In particular,

$b_k(N) = b_k(M)$

for all $k$, and

$\chi(N) = \chi(M)$.

Remark. For any smooth map $\varphi : M \to N$, The cup product makes

$H^*_dR(M) = \bigoplus_{k=0}^m H^k_{dR}(M)$

a graded ring, and the induced map $\varphi^*$ is in fact a ring homomorphism

$\varphi^* : H^*_dR(N) \to H^*_dR(M)$,

since

$\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta$.

Moreover, if $\varphi$ is a diffeomorphism, then $\varphi^* : H^*_dR(N) \to H^*_dR(M)$ is a ring isomorphism.

2. Homotopic Invariance

¶ Homotopic Invariance of de Rham cohomology.

In this section we shall prove a much stronger result: if two manifolds are homotopy equivalent, then they have the same de Rham cohomology groups. We first recall

Definition 2.1. Two topological spaces $M$ and $N$ are said to be homotopy equivalent if there exist continuous maps $\varphi : M \to N$ and $\psi : N \to M$ so that $\varphi \circ \psi$ is homotopic to $\text{Id}_N$ and $\psi \circ \varphi$ is homotopic to $\text{Id}_M$.

In Lecture 11 we have shown
• any continuous map between smooth manifolds is homotopic to some smooth map.
• any two homotopic smooth maps are smoothly homotopic (i.e., the homotopy $F$ can be chosen to be smooth if both $f_0$ and $f_1$ are smooth).

Homotopy equivalence is a much weaker equivalence relation than homeomorphism or diffeomorphism. For example (check details if you are not familiar with this)
• $S^{n-1}$ is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$.
• any star-shaped region is homotopic equivalent to a single point set $\{x_0\}$.

We would like to prove

**Theorem 2.2** (Homotopy Invariance). Let $M, N$ be smooth manifolds. If $M$ and $N$ are homotopy equivalent, then

$$H^k_{dR}(M) \simeq H^k_{dR}(N), \quad \forall k.$$  

Obviously the homotopy invariance implies the topological Invariance of de Rham cohomology groups:

If $M$ is homeomorphic to $N$, then $H^k_{dR}(M) \simeq H^k_{dR}(N)$ for all $k$.

**Remark.** Although in defining $H^k_{dR}(M)$, we need to use the smooth structure on $M$ (to define the operator $d$ and the space $\Omega^k(M)$ etc), the topological invariance tells us that $H^k_{dR}(M)$ only depends on the topology of $M$, and is independent of the smooth structure! In fact, for any topological space $X$, one can define its *singular cohomology groups* $H^k_{sing}(X, \mathbb{R})$ which depends only on the topology (and in fact depends only on the homotopy class) of $X$. The famous theorem of de Rham claims

**Theorem 2.3** (The de Rham theorem). $H^k_{dR}(M) = H^k_{sing}(M, \mathbb{R})$ for all $k$.

We will not prove the theorem in this course.

Another immediate consequence of the homotopy invariance is

**Corollary 2.4** (Poincare’s lemma). If $U$ is a star-shaped region in $\mathbb{R}^m$, then for any $k \geq 1$, $H^k_{dR}(U) = 0$. In particular,

$$H^k_{dR}(\mathbb{R}^m) = 0, \quad \forall k \geq 1.$$  

Since any point in a manifold has a neighborhood that is homeomorphic to a star-like region in $\mathbb{R}^n$, we conclude that any closed form is locally exact:

**Corollary 2.5.** Suppose $k \geq 1$. Then for any closed $k$-form $\omega \in \Omega^k(M)$ and any $p \in M$, there is a neighborhood $U$ of $p$ and an $(k-1)$-form $\eta \in \Omega^{k-1}(U)$ so that

$$\omega = d\eta$$  

on $U$.  

Then there is a linear operator \( \phi_1 \) on the existence of cochain homotopy. Homotopic invariance of de Rham cohomology: The proof.

**Theorem 2.6.** Let \( f, g \in C^\infty(M, N) \) be homotopic, then

\[
f^* = g^* : H^k_{dR}(N) \to H^k_{dR}(M).
\]

**Proof of Theorem 2.2 assuming Theorem 2.6.** Let \( \varphi : M \to N \) and \( \psi : N \to M \) be continuous maps so that \( \varphi \circ \psi \sim \text{Id}_N \) and \( \psi \circ \varphi \sim \text{Id}_M \). Then one can find \( \varphi_1 \in C^\infty(M, N) \) and \( \psi_1 \in C^\infty(N, M) \) so that \( \varphi_1 \sim \varphi \) and \( \psi_1 \sim \psi \). It follows that both \( \varphi_1 \circ \psi_1 \) and \( \psi_1 \circ \varphi_1 \) are smooth, and \( \varphi_1 \circ \psi_1 \sim \text{Id}_N, \psi_1 \circ \varphi_1 \sim \text{Id}_M \).

Applying Theorem 2.6, we get

\[
\varphi_1^* \circ \psi_1^* = \text{Id} : H^k_{dR}(M) \to H^k_{dR}(M)
\]

\[
\psi_1^* \circ \varphi_1^* = \text{Id} : H^k_{dR}(N) \to H^k_{dR}(N).
\]

So \( \varphi^* \) and \( \psi^* \) are linear isomorphisms. \( \square \)

Theorem 2.6 can be proved by constructing a cochain homotopy:

\[
\cdots \to \Omega^{k-1}(N) \xrightarrow{d} \Omega^k(N) \xrightarrow{d} \Omega^{k+1}(N) \xrightarrow{d} \cdots
\]

\[
\xymatrix{ & \Omega^{k-1}(M) \ar[rr]^d \ar[ld]^{h_k \circ f^*} \ar[dd]^{g^*} & & \Omega^k(M) \ar[rr]^d \ar[ld]^{h_k \circ f^*} \ar[dd]^{g^*} & & \Omega^{k+1}(M) \ar[ld]^{h_k \circ f^*} \ar[dd]^{g^*} & \cdots}
\]

**Definition 2.7.** Let \( f, g \in C^\infty(M, N) \) be homotopic. A cochain homotopy between \( f^* \) and \( g^* \) is a sequence of maps \( h_k : \Omega^k(N) \to \Omega^{k-1}(M) \) such that on \( \Omega^k(N) \),

\[
g^* - f^* = d_M h_k + h_{k+1} d_N.
\]

**Proof of Theorem 2.6 assuming the existence of the cochain homotopy.**

Suppose \( [\omega] \in H^k_{dR}(N) \). Then \( d\omega = 0 \) since \( \omega \) is closed. It follows

\[
g^* \omega - f^* \omega = (dh + hd)\omega = dh\omega \in B^k(M)
\]

Thus \( f^*([\omega]) = [f^* \omega] = [g^* \omega] = g^*([\omega]) \). \( \square \)

**The existence of cochain homotopy.**

It remains to construct the cochain homotopy. We will use the flow generated by a vector field to complete the construction. Recall that if \( X \) is a complete vector field on \( M \), then \( X \) generates a flow \( \phi_t : M \to M \). We need

**Lemma 2.8.** Let \( X \) be a complete vector field on \( M \), and \( \phi_t \) the flow generated by \( X \). Then there is a linear operator \( Q : \Omega^k(M) \to \Omega^{k-1}(M) \) so that for any \( \omega \in \Omega^k(M) \),

\[
\phi_t^* \omega - \omega = dQ(\omega) + Q(d\omega).
\]
Proof. If we set $Q_t(\omega) = \iota_X(\phi_t^*\omega)$, then $Q_t : \Omega^k(M) \to \Omega^{k-1}(M)$ and
\[
\frac{d}{dt} \phi_t^*\omega = \left. \frac{d}{ds} \phi_{t+s}^*\omega \right|_{s=0} = \mathcal{L}_X(\phi_t^*\omega) = d\iota_X(\phi_t^*\omega) + \iota_X(d(\phi_t^*\omega)) = d(Q_t\omega) + \iota_X(\phi_t^*(d\omega)) = d(Q_t\omega) + Q_t(d\omega).
\]
So if we denote $Q(\omega) = \int_0^1 Q_t(\omega) dt$, then $Q : \Omega^k(M) \to \Omega^{k-1}(M)$ and
\[
\phi_t^*\omega - \omega = \int_0^1 \left( \frac{d}{dt} \phi_t^*\omega \right) dt = d(Q(\omega)) + Q(d\omega). \quad \square
\]

**Construction of the cochain homotopy $h_k : \Omega^k(N) \to \Omega^{k-1}(M)$**.

Let $W = M \times \mathbb{R}$, then $X = \frac{\partial}{\partial t}$ is a complete vector field on $W$ whose flow is $\phi_t(p, a) = (p, a + t)$.

By Lemma 2.8, there is a linear operator $Q : \Omega^k(W) \to \Omega^{k-1}(W)$ so that
\[
\phi_t^*\omega - \omega = dQ(\omega) + Q(d\omega).
\]

Let $F : W \to N$ is a smooth homotopy between $f$ and $g$, and let $\iota : M \hookrightarrow W$ be the inclusion map $\iota(p) = (p, 0)$, then
\[
f = F \circ \iota \quad \text{and} \quad g = F \circ \phi_1 \circ \iota,
\]
It follows that for any $\omega \in \Omega^k(N)$,
\[
g^*\omega - f^*\omega = \iota^*\phi_t^*F^*\omega - \iota^*F^*\omega = \iota^*(dQ + Qd)F^*\omega = (d\iota^*QF^* + \iota^*QF^*d)\omega.
\]
So if we denote $h = \iota^*QF^*$, then $h : \Omega^k(N) \to \Omega^{k-1}(M)$ satisfies
\[
g^*\omega - f^*\omega = (dh + hd)\omega,
\]
i.e. $h$ is the cochain homotopy we are looking for. \quad \square