

LECTURE 26: THE MAYER-VIETORIS SEQUENCE

The Mayer-Vietoris sequence is an algebraic tool using which one can compute homology/cohomology groups of topological spaces from the knowledge of the homology/cohomology groups of suitable subspaces. It is a sort of analogue of the (Seifert-)van Kampen theorem for the fundamental group. Moreover, partly because of the cohomology groups are simpler (abelian) than the fundamental groups, the assumption for the Mayer-Vietoris sequence is weaker, for example, no path-connectedness is needed.

1. THE MAYER-VIETORIS SEQUENCE

¶ Exact sequences.

Suppose we have a *cochain complex* (\mathcal{A}, d) , i.e. a sequence

$$\dots \xrightarrow{d_{k-2}} A^{k-1} \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} A^{k+2} \xrightarrow{d_{k+2}} \dots$$

where A^k 's are vector spaces, and d_k 's are linear maps such that

$$d_k \circ d_{k-1} = 0, \quad \forall k.$$

Obviously we have $\text{Im}(d_k) \subset \text{Ker}(d_{k+1})$. One can define cohomology groups for any cochain complex as we did for the de Rham cochain complex:

$$H^k(\mathcal{A}) := \ker(d_k) / \text{Im}(d_{k-1})$$

Such a complex (\mathcal{A}, d) is called an *exact sequence* if $H^k(\mathcal{A}, d) = 0$ for all k , i.e.

$$\text{Im}(d_{k-1}) = \ker(d_k), \quad \forall k.$$

Note that if an exact sequence starts with 0,

$$(\dots \rightarrow) 0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \xrightarrow{d_3} \dots,$$

then $d_1 : V^1 \rightarrow V^2$ is injective, while if an exact sequence ends with 0,

$$\dots \xrightarrow{d_{k-2}} V^{k-1} \xrightarrow{d_{k-1}} V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} 0 \quad (\rightarrow \dots),$$

then $d_k : V^k \rightarrow V^{k+1}$ is surjective. In particular, if we have a *short exact sequence*

$$0 \longrightarrow V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \longrightarrow 0,$$

then d_1 is injective, d_2 is surjective, and

$$V^2 \simeq \ker(d_2) \oplus \text{Im}(d_2) \simeq \text{Im}(d_1) \oplus \text{Im}(d_2) \simeq V^1 \oplus V^3.$$

Another useful fact about exact sequences is: if a finite sequence

$$0 \rightarrow A^1 \rightarrow A^2 \rightarrow \dots \rightarrow A^k \rightarrow 0$$

is exact, then (The proof is left as an exercise.)

$$\sum_i (-1)^i \dim A^i = 0.$$

¶ **Short exact sequence of complexes.**

A general principle in homological algebra: Given three cochain complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$, which form a short exact sequence $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ in the sense that for any k ,

$$0 \rightarrow A^k \rightarrow B^k \rightarrow C^k \rightarrow 0$$

is a short exact sequence, then one can construct a long exact sequence of the cohomology groups

$$\dots \rightarrow H^{k-1}(\mathcal{C}) \rightarrow H^k(\mathcal{A}) \rightarrow H^k(\mathcal{B}) \rightarrow H^k(\mathcal{C}) \rightarrow H^{k+1}(\mathcal{A}) \rightarrow \dots$$

Now suppose M is a smooth manifold, and U, V are open sets in M so that $M = U \cup V$. Since M, U, V and $U \cap V$ are all smooth manifolds, we have four de Rham complexes,

$$\Omega^*(M) : \quad 0 \rightarrow \Omega^0(M) \rightarrow \Omega^1(M) \rightarrow \Omega^2(M) \rightarrow \Omega^3(M) \rightarrow \dots$$

$$\Omega^*(U) : \quad 0 \rightarrow \Omega^0(U) \rightarrow \Omega^1(U) \rightarrow \Omega^2(U) \rightarrow \Omega^3(U) \rightarrow \dots$$

$$\Omega^*(V) : \quad 0 \rightarrow \Omega^0(V) \rightarrow \Omega^1(V) \rightarrow \Omega^2(V) \rightarrow \Omega^3(V) \rightarrow \dots$$

and

$$\Omega^*(U \cap V) : \quad 0 \rightarrow \Omega^0(U \cap V) \rightarrow \Omega^1(U \cap V) \rightarrow \Omega^2(U \cap V) \rightarrow \Omega^3(U \cap V) \rightarrow \dots$$

It turns out that these complexes form a short exact sequence

$$0 \rightarrow \Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V) \rightarrow 0.$$

To see this, we consider the inclusion maps

$$\iota_1 : U \hookrightarrow M, \quad \iota_2 : V \hookrightarrow M$$

and the inclusion maps

$$j_1 : U \cap V \hookrightarrow U, \quad j_2 : U \cap V \hookrightarrow V.$$

These inclusion maps induce linear maps between the spaces of k -forms (and also induce linear maps between the corresponding de Rham cohomology groups, which we use the same notation)

$$\alpha_k : \Omega^k(M) \rightarrow \Omega^k(U) \oplus \Omega^k(V), \quad \omega \mapsto (\iota_1^* \omega, \iota_2^* \omega)$$

and

$$\beta_k : \Omega^k(U) \oplus \Omega^k(V) \rightarrow \Omega^k(U \cap V), \quad (\omega_1, \omega_2) \mapsto j_1^* \omega_1 - j_2^* \omega_2.$$

The short exact sequence we alluded above is stated in the following proposition. The proof is simple and is left as an exercise.

Proposition 1.1. *For any k , the sequence*

$$0 \longrightarrow \Omega^k(M) \xrightarrow{\alpha_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta_k} \Omega^k(U \cap V) \longrightarrow 0$$

is a short exact sequence.

¶ **From short exact sequence of complexes to long exact sequence.**

According to the general principle in homological algebra that we mentioned above, we should be able to construct a long exact sequence consisting of corresponding de Rham cohomology groups. Let's explain the construction in our setting in detail.

The maps α_k and β_k defined above induces maps

$$\alpha_k : H_{dR}^k(M) \rightarrow H_{dR}^k(U) \oplus H_{dR}^k(V)$$

and

$$\beta_k : H_{dR}^k(U) \oplus H_{dR}^k(V) \rightarrow H_{dR}^k(U \cap V).$$

To get a long exact sequence, we need to define a linear map (called the *connecting homomorphism*)

$$\delta_k : H_{dR}^k(U \cap V) \rightarrow H_{dR}^{k+1}(M).$$

The map δ_k can be defined by the standard “diagram chasing” method. In what follows we will give an explicit construction: We fix a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the cover $\{U, V\}$ of M . For any $\omega \in Z^k(U \cap V)$, we define

$$\delta_k([\omega]) := [\eta],$$

where η is the $(k+1)$ -form

$$\eta = \begin{cases} d(\rho_V \omega) & \text{on } U, \\ -d(\rho_U \omega) & \text{on } V. \end{cases}$$

(Why should we define η like this? Well, we need a $(k+1)$ -form on M . The only way to get such a smooth $(k+1)$ -form on M is to take the exterior derivative of a smooth k -form on M . Unfortunately the zero extension of ω to M is not a smooth k -form on M since ω could be “nonzero near the boundary of $U \cap V$. P.O.U is the correct way to “force ω to be zero near the boundaries”!)

Lemma 1.2. *The map δ_k is well-defined.*

Proof. There are many issues to be checked:

- $\rho_V \omega \in \Omega^k(U)$: since $\rho_V \in C^\infty(M)$ and $\text{supp}(\rho_V) \cap U \subset U \cap V$. (Note: (1) ω is NOT defined on $U \setminus U \cap V$. (2) $\rho_V \omega \notin \Omega^k(M)$.) Similarly $\rho_U \omega \in \Omega^k(V)$.
- $\eta \in \Omega^{k+1}(M)$: Since $\rho_U + \rho_V = 1$ and since $d\omega = 0$ on $U \cap V$, we see

$$d(\rho_V \omega) = -d(\rho_U \omega)$$

on $U \cap V$, so η is a well-defined smooth $(k+1)$ -form on M .

- $\eta \in Z^{k+1}(M)$: By definition we have $d\eta = 0$ on both U and V . So $\eta \in Z^{k+1}(M)$.
- $[\eta]$ is independent of the choices of ρ_U and ρ_V : Let $\tilde{\rho}_U$ and $\tilde{\rho}_V$ be another partition of unity subordinate to the cover $\{U, V\}$, and let $\tilde{\eta}$ be the resulting $(k+1)$ -form. Then $\tilde{\rho}_V - \rho_V = \rho_U - \tilde{\rho}_U$ is supported in $U \cap V$. So if we let

$$\xi = (\tilde{\rho}_V - \rho_V)\omega,$$

then it is a smooth k -form on M , and by the construction, we have $\tilde{\eta} - \eta = d\xi$ on both U and V , and thus on M .

- $[\eta]$ is independent of the choices of ω : Suppose $\tilde{\omega} = \omega + d\zeta$ and denote the resulting $(k+1)$ -form by $\tilde{\eta}$. Then

$$\tilde{\eta} - \eta = \begin{cases} d(\rho_V d\zeta) & \text{on } U, \\ -d(\rho_U d\zeta) & \text{on } V. \end{cases}$$

We define

$$\xi = \begin{cases} -d\rho_V \wedge \zeta & \text{on } U, \\ d\rho_U \wedge \zeta & \text{on } V. \end{cases}$$

Then by the same argument above, ξ is a smooth k -form on M . Moreover,

$$d\xi = \begin{cases} d\rho_V \wedge d\zeta = d(\rho_V d\zeta) & \text{on } U \\ -d\rho_U \wedge d\zeta = -d(\rho_U d\zeta) & \text{on } V \end{cases} = \tilde{\eta} - \eta.$$

So $[\tilde{\eta}] = [\eta]$ and the conclusion follows. \square

¶ The Mayer-Vietoris Theorem.

Now we can state the main theorem:

Theorem 1.3 (Mayer-Vietoris). *Let U, V be open sets in M so that $M = U \cup V$. Then we have a long exact sequence*

$$\dots \xrightarrow{\delta_{k-1}} H_{dR}^k(M) \xrightarrow{\alpha_k} H_{dR}^k(U) \oplus H_{dR}^k(V) \xrightarrow{\beta_k} H_{dR}^k(U \cap V) \xrightarrow{\delta_k} H_{dR}^{k+1}(M) \xrightarrow{\alpha_{k+1}} \dots$$

Proof. One has to show

- $\text{Im}(\alpha_k) = \ker(\beta_k)$,
- $\text{Im}(\beta_k) = \ker(\delta_k)$,
- $\text{Im}(\delta_k) = \ker(\alpha_{k+1})$.

This amounts to prove 6 inclusion relations. We will prove one of them and leave the rest as exercises.

Proof of $\text{Im}(\beta_k) \subset \ker(\delta_k)$: Suppose $\omega_1 \in Z^k(U)$, $\omega_2 \in Z^k(V)$. Let

$$\omega := \beta_k(\omega_1, \omega_2) = j_1^* \omega_1 - j_2^* \omega_2 \in \Omega^k(U \cap V).$$

Then $\delta_k([\omega]) = [\eta]$, where

$$\eta = \begin{cases} d(\rho_V \omega) = d(\rho_V \omega - \omega_1) & \text{on } U, \\ -d(\rho_U \omega) = -d(\rho_U \omega + \omega_2) & \text{on } V. \end{cases}$$

Note that on $U \cap V$,

$$\rho_V \omega - j_1^* \omega_1 = -\rho_U \omega - j_2^* \omega_2.$$

So there is a smooth k -form ξ on M so that

$$\xi = \begin{cases} \rho_V \omega - \omega_1 & \text{on } U, \\ -\rho_U \omega - \omega_2 & \text{on } V. \end{cases}$$

As a consequence, $\eta = d\xi$ and thus $[\eta] = 0$. \square

Remark. As in the case of van Kampen's theorem, in general knowing all $H_{dR}^k(U)$, $H_{dR}^k(V)$ and $H_{dR}^k(U \cap V)$'s are not enough to determine the $H_{dR}^k(M)$'s. To determine the $H_{dR}^k(M)$'s we also need the homomorphisms connecting them.

2. APPLICATIONS OF MAYER-VIETORIS SEQUENCE

¶ **Application 1: The de Rham cohomology groups of the spheres.**

Theorem 2.1. For $n \geq 1$, $H_{dR}^k(S^n) \simeq \begin{cases} \mathbb{R}, & k = 0, n, \\ 0, & 1 \leq k \leq n-1. \end{cases}$

Proof. Last time we have proven

$$H_{dR}^0(S^n) \simeq \mathbb{R} \quad \text{and} \quad H_{dR}^1(S^1) \simeq \mathbb{R}.$$

In what follows we will prove

- (1) For $n \geq 2$, $H_{dR}^1(S^n) = 0$.
- (2) For $n \geq 2, k \geq 2$, $H_{dR}^k(S^n) \simeq H_{dR}^{k-1}(S^{n-1})$.

Obviously these results together imply the theorem.

For $n \geq 2$, we let

$$U = S^n - \{(0, \dots, 0, -1)\} \quad \text{and} \quad V = S^n - \{(0, \dots, 0, 1)\}.$$

Then

- $M = U \cup V$,
- U and V are diffeomorphic to \mathbb{R}^n ,
- $U \cap V$ is homotopy equivalent to S^{n-1} .

To prove (1), we look at the beginning of the Mayer-Vietories sequence

$$0 \longrightarrow H_{dR}^0(S^n) \longrightarrow H_{dR}^0(U) \oplus H_{dR}^0(V) \longrightarrow H_{dR}^0(U \cap V) \longrightarrow H^1(S^n) \rightarrow H_{dR}^1(U) \oplus H_{dR}^1(V),$$

which now becomes

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha_0} \mathbb{R}^2 \xrightarrow{\beta_0} \mathbb{R} \xrightarrow{\delta_0} H_{dR}^1(S^n) \rightarrow 0.$$

Since α_0 is injective,

$$\dim \ker(\beta_0) = \dim \text{Im}(\alpha_0) = 1.$$

It follows that

$$\dim \text{Im}(\beta_0) = \dim \mathbb{R}^2 - \dim \ker(\beta_0) = 1,$$

i.e. β_0 is surjective. So $\ker(\delta_0) = \mathbb{R}$, i.e. $\delta_0 \equiv 0$. But by exactness, δ_0 is surjective. This implies $H_{dR}^1(S^n) = 0$.

To prove (2), we look at the following part of the Mayer-Vietories sequence

$$H_{dR}^{k-1}(U) \oplus H_{dR}^{k-1}(V) \xrightarrow{\beta_{k-1}} H_{dR}^{k-1}(U \cap V) \xrightarrow{\delta_{k-1}} H_{dR}^k(S^n) \xrightarrow{\alpha_k} H_{dR}^k(U) \oplus H_{dR}^k(V),$$

which becomes

$$0 \xrightarrow{\beta_{k-1}} H_{dR}^{k-1}(S^{n-1}) \xrightarrow{\delta_{k-1}} H_{dR}^k(S^n) \xrightarrow{\alpha_k} 0.$$

By exactness, the map δ_{k-1} has to be both injective and surjective, and thus must be a linear isomorphism. This proves (2). \square

As a consequence, we get a simple proof of the topological invariance of the dimension:

Corollary 2.2 (Topological Invariance of Dimension). *If $m \neq n$, then \mathbb{R}^n is not homeomorphic to \mathbb{R}^m .*

Proof. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a homeomorphism, then $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{f(0)\}$ is a homeomorphism. So

$$H_{dR}^k(\mathbb{R}^n \setminus \{0\}) = H_{dR}^k(\mathbb{R}^m \setminus \{f(0)\}), \quad \forall k.$$

But $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1} , while $\mathbb{R}^m \setminus \{f(0)\}$ is homotopic equivalent to S^{m-1} . So

$$H_{dR}^k(S^{m-1}) = H_{dR}^k(S^{n-1}), \quad \forall k.$$

This contradicts with the fact $m \neq n$. □

¶ **Application 2:** $\dim H_{dR}^k(M) < \infty$ for many smooth manifolds.

Definition 2.3. Let M be a smooth manifold and $\{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of M . We say $\{U_\alpha\}_{\alpha \in \Lambda}$ is a *good cover* if for any finite subset $I = \{\alpha_1, \dots, \alpha_k\} \subset \Lambda$ of indices, the intersection

$$U_I := U_{\alpha_1} \cap U_{\alpha_2} \cap \dots \cap U_{\alpha_k}$$

is either empty or diffeomorphic to \mathbb{R}^n .

Example. The cover of S^n by $S^n - \{N\}$ and $S^n - \{S\}$ is not a good cover. However, the cover of S^n by the $2n + 2$ hemispheres

$$S_{i,+(-)}^n = \{(x^1, \dots, x^{n+1}) \mid x^i > (<)0\}$$

is a good cover (why?).

By using Riemannian geometry (more precisely, use the so-called geodesically convex neighborhoods.), one can show that any open cover of any smooth manifold M admits a refinement which is a good cover. In particular, if M is compact, then M admits a good cover which contains only finitely many sets. Such a cover is called a *finite good cover*.

Obviously any sub-cover of a good cover is still a good cover.

Theorem 2.4. *If M admits a finite good cover, then $H_{dR}^k(M)$ is finite dimensional.*

Proof. We proceed by induction on the number of sets in a finite good cover of M . If M admits a good cover that contains only one open set, then that open set has to be M itself. In this case M is diffeomorphic to \mathbb{R}^n and the conclusion follows.

Now suppose the theorem holds for any manifold that admits a good cover containing $l - 1$ open sets. Let M be a manifold with an good cover $\{U_1, \dots, U_l\}$. We denote

$$U = U_1 \cup \dots \cup U_{l-1} \quad \text{and} \quad V = U_l.$$

Then $U \cap V$ admits a finite good cover $\{U_1 \cap U_l, \dots, U_{l-1} \cap U_l\}$. By the induction hypothesis, all the de Rham cohomology groups of U , V and $U \cap V$ are finite dimensional. Now consider the Mayer-Vietoris sequence

$$\dots \longrightarrow H_{dR}^{k-1}(U \cap V) \xrightarrow{\delta_{k-1}} H_{dR}^k(M) \xrightarrow{\alpha_k} H_{dR}^k(U) \oplus H_{dR}^k(V) \longrightarrow \dots$$

The conclusion follows since

$$\dim \operatorname{Im}(\alpha_k) \leq \dim H_{dR}^k(U) \oplus H_{dR}^k(V) < \infty$$

and

$$\dim \ker(\alpha_k) = \dim \operatorname{Im}(\delta_{k-1}) \leq \dim H_{dR}^{k-1}(U \cap V) < \infty. \quad \square$$

As a consequence, we immediately get

Corollary 2.5. *If M is compact or M is homotopy equivalent to a compact manifold, then $\dim H_{dR}^k(M) < \infty$ for all k .*

¶ Application 3: The Kunneth formula.

Theorem 2.6. *Let M and N be manifolds with finite good covers. Then for any $0 \leq k \leq \dim M + \dim N$, one has*

$$H_{dR}^k(M \times N) \simeq \bigoplus_{i=0}^k H_{dR}^i(M) \otimes H_{dR}^{k-i}(N).$$

Sketch of proof. Let $\pi_M : M \times N \rightarrow M$ and $\pi_N : M \times N \rightarrow N$ be the standard projections. Then we get a map

$$\Psi : \Omega^*(M) \otimes \Omega^*(N) \rightarrow \Omega^*(M \times N), \quad \omega_1 \otimes \omega_2 \mapsto \pi_M^* \omega_1 \wedge \pi_N^* \omega_2.$$

One can check that this map induces a map on cohomologies

$$\Psi : H_{dR}^*(M) \otimes H_{dR}^*(N) \rightarrow H_{dR}^*(M \times N), \quad [\omega_1] \otimes [\omega_2] \mapsto [\pi_M^* \omega_1 \wedge \pi_N^* \omega_2].$$

To prove that this Ψ is in fact an linear isomorphism, we do induction on the number l of elements in a good cover of M .

If $l = 1$, i.e. M is diffeomorphic to \mathbb{R}^n , then the Kunneth formula follows from the fact that $\mathbb{R}^n \times N$ is homotopy equivalent to N , and $H_{dR}^k(\mathbb{R}^n) = \mathbb{R}$ for $k = 0$ and $H_{dR}^k(\mathbb{R}^n) = 0$ for other k 's.

Now suppose the Kunneth formula is proved for manifolds admitting a good cover with no more than $l - 1$ open sets, and suppose now that $M = U_1 \cup \dots \cup U_l$ is a good cover. Again we let $U = U_1 \cup \dots \cup U_{l-1}$ and $V = U_l$. For simplicity we will denote

$$\tilde{H}^k(M, N) := \bigoplus_{i=0}^k H_{dR}^i(M) \otimes H_{dR}^{k-i}(N).$$

Consider the following diagram

$$\begin{array}{ccccccc} \tilde{H}^k(M, N) & \xrightarrow{\alpha} & \tilde{H}^k(U, N) \oplus \tilde{H}^k(V, N) & \xrightarrow{\beta} & \tilde{H}^k(U \cap V, N) & \xrightarrow{\delta} & \tilde{H}^{k+1}(M, N) \\ \Psi \downarrow & & \Psi \downarrow & & \Psi \downarrow & & \Psi \downarrow \\ H_{dR}^k(M \times N) & \xrightarrow{\alpha} & H_{dR}^k(U \times N) \oplus H_{dR}^k(V \times N) & \xrightarrow{\beta} & H_{dR}^k((U \cap V) \times N) & \xrightarrow{\delta} & H_{dR}^{k+1}(M \times N) \end{array}$$

where the horizontal maps α, β, δ 's are the ones that is induced in the obvious way from the $\alpha_k, \beta_k, \delta_k$'s that we defined above. For this diagram we have

- By using the Mayer-Vietoris sequence one can prove the two rows are exact.

- Moreover, one can prove that the diagram is commutative, i.e. we have $\Psi \circ \alpha = \alpha \circ \Psi$, $\Psi \circ \beta = \beta \circ \Psi$ and $\Psi \circ \delta = \delta \circ \Psi$. [The first two equalities are easy to prove, while the last one is much more complicated.]
- By using the induction hypothesis, the second and the third Ψ are linear isomorphisms.

Now the result follows from the well-known Five lemma in homological algebra. \square

Lemma 2.7 (Five lemma). *Suppose we have the following commutative diagram*

$$\begin{array}{ccccccccc}
 V_1 & \xrightarrow{\alpha_1} & V_2 & \xrightarrow{\alpha_2} & V_3 & \xrightarrow{\alpha_3} & V_4 & \xrightarrow{\alpha_4} & V_5 \\
 \gamma_1 \downarrow & & \gamma_2 \downarrow & & \gamma_3 \downarrow & & \gamma_4 \downarrow & & \gamma_5 \downarrow \\
 W_1 & \xrightarrow{\beta_1} & W_2 & \xrightarrow{\beta_2} & W_3 & \xrightarrow{\beta_3} & W_4 & \xrightarrow{\beta_4} & W_5
 \end{array}$$

where each V_i and each W_i is a linear space, and each map is a linear map. Suppose the two rows are exact, and the vertical maps $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ are isomorphisms, then the map γ_3 is also an isomorphism.

Proof. This is a standard diagram-chasing exercise. \square

As a simple consequence of the Kunneth formula, we can compute

$$H_{dR}^k(S^n \times S^n) \simeq \begin{cases} \mathbb{R}, & k = 0 \text{ or } 2n, \\ \mathbb{R}^2, & k = n, \\ 0, & \text{other } k. \end{cases}$$

Similarly one can calculate $H_{dR}^k(S^m \times S^n)$ for $m \neq n$.

We can also compute the de Rham cohomology groups of the n -torus \mathbb{T}^n .

Proposition 2.8. *For the n -dimensional torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$,*

$$H_{dR}^k(\mathbb{T}^n) \simeq \mathbb{R}^{\binom{n}{k}}.$$

The proof is left as an exercise.