LECTURE 28: APPLICATIONS OF DE RHAM THEORY

1. The mapping degree and its applications

¶ The mapping degree.

Let M, N be m-dimensional connected oriented manifolds, and $f : M \to N$ a proper smooth map. Then the pull-back map

$$f^* : \mathbb{R} \simeq H_c^m(N) \longrightarrow H_c^m(M) \simeq \mathbb{R}$$

is linear, and thus is a map of the form $\lambda \mapsto c\lambda$. The constant c is called the degree of f:

Definition 1.1. The *degree* of a proper smooth map $f : M \to N$ is the number $\deg(f)$ s.t.

$$\int_M f^*\omega = \deg(f) \int_N \omega, \qquad \forall \omega \in \Omega_c^m(N).$$

Remark. The isomorphism $H_c^m(N) \simeq \mathbb{R}$ is induced by the integral $\int_N$ and thus does depend on the choices of orientations: with respect to opposite orientations, the same ω will give us opposite real numbers. Thus to define $\deg(f)$ we need to fix orientations on M and N first. (However, if M is orientable, and $f : M \to M$ is a map from M to itself, then $\deg(f)$ is independent of the choices of the orientations on M.)

We have

Proposition 1.2. *Let M, N, P be connected oriented manifolds of the same dimension.*

(1) *If $f : M \to N$ and $g : N \to P$ are both proper smooth maps, then $g \circ f$ is proper, and*

$$\deg(g \circ f) = \deg(f)\deg(g).$$

(2) *If $f : M \to N$ and $g : M \to N$ are properly homotopic maps , then* $\deg(f) = \deg(g)$.

(3) *If $f : M \to N$ is a diffeomorphism, then*

$$\deg(f) = \begin{cases} 1, & f \text{ is orientation preserving,} \\ -1, & f \text{ is orientation reversing.} \end{cases}$$

Proof. (1) follows from definition, (2) follows from the fact $f^* = g^*$ on compactly supported cohomology groups, and (3) follows from the change of variable formula. □

Remark. We have seen that any continuous map is homotopic to a smooth map, and homotopic smooth maps are smoothly homotopic. Moreover, under the compactness assumption, any homotopy is a proper homotopy. So if the manifolds under consideration are compact, we can define the degree of a continuous map to be the degree of the corresponding smooth maps, which is independent of the choices of the smooth approximation. In fact, in algebraic topology, the mapping degree theory is developed for continuous maps: in that case there is no smoothness at all.

¶ Computing the mapping degree.

We give a couple examples to compute the mapping degree.

Example. Consider the antipodal map

$$f : S^n \to S^n, \qquad p \mapsto f(p) = -p.$$

Then according to PSet 7-1-5(e),

$$\deg(f) = \begin{cases} 1, & n \text{ is odd,} \\ -1, & n \text{ is even.} \end{cases}$$

It turns out that $\deg(f) = 0$ as long as f is not surjective:

Proposition 1.3. *If a proper smooth map $f : M \to N$ is not surjective, then* $\deg(f) = 0$.

Proof. According to PSet 3-1-5(a), f is a closed map. So if $q \notin f(M)$, then one can find an open neighborhood $\widetilde{U}$ of q so that $\widetilde{U} \cap f(M) = \emptyset$. We pick an m-form ω supported in $\widetilde{U}$ so that $\int_N \omega = 1$. Since by our construction, $f^*\omega = 0$, we conclude $\deg(f) = 0$. □

In fact, we have the following remarkable property of the degree:

$$\boxed{\deg(f) \text{ is always an integer!}}$$

To see this, we may assume that f is surjective. Let $q \in N$ be a regular value of f. According to PSet 3-1-5(c), $f^{-1}(q) = \{p_1, \cdots, p_k\}$ is a finite set, and there exists a neighborhood $\widetilde{U}$ of q and neighborhoods U_i of each p_i, so that

- For $i \neq j$, $U_i \cap U_j = \emptyset$, and $f^{-1}(\widetilde{U}) = \cup_{i=1}^k U_i$.
- f maps U_i diffeomorphically to $\widetilde{U}$.

Obviously we can choose U and U_i's to be small connected oriented charts. We let

$$\sigma_i = \begin{cases} 1, & \text{if } f : U_i \to \widetilde{U} \text{ is orientation preserving,} \\ -1, & \text{if } f : U_i \to \widetilde{U} \text{ is orientation reversing.} \end{cases}$$

The fact $\deg(f)$ is an integer follows from

Theorem 1.4. *The degree of f is the integer*

$$\deg(f) = \sum_{i=1}^k \sigma_i.$$

Proof. Take $\omega \in \Omega_c^n(\widetilde{U})$ so that $\int_N \omega = 1$. Then $f^*\omega$ is supported in $f^{-1}(\widetilde{U}) = \cup_{i=1}^k U_i$, and

$$\int_M f^*\omega = \sum_{i=1}^k \int_{U_i} f^*\omega = \sum_{i=1}^k \sigma_i \int_{\widetilde{U}} \omega = \sum_{i=1}^k \sigma_i.$$

□

Example. The degree of the map

$$f : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto f(z) = z^k$$

is $\deg(f) = k$.

¶ Application 1: The Hairy Ball Theorem.

Theorem 1.5. *Even dimensional spheres do not admit non-vanishing smooth vector fields.*

Proof. Suppose X is a non-vanishing smooth vector field on $S^{2n} \subset \mathbb{R}^{2n+1}$. By normalizing the vectors, we may assume $|X_p| = 1$ for all $p \in S^{2n}$. We will think of p and X_p as vectors in $\mathbb{R}^{2n+1}$, and consider the map $F : S^{2n} \times [0,1] \to S^{2n}$ defined by

$$F(p,t) = p\cos(t\pi) + X_p \sin(t\pi).$$

Then for each $t \in [0,1]$, $F(\cdot,t)$ is a map from S^{2n} to S^{2n}. (Note: $|p| = |X_p| = 1$ and $p \perp X_p$.) So F is a homotopy between the identity map $F(\cdot,0) = \mathrm{Id}_{S^{2n}}$ and the antipodal map $F(\cdot,1) = f : S^{2n} \to S^{2n}, f(x) = -x$. So $-1 = \deg(f) = \deg(\mathrm{Id}_{S^{2n}}) = 1$, a contradiction. □

As a consequence, we get

Corollary 1.6. *For any $n \geq 1$, S^{2n} admits no Lie group structure.*

¶ Application 2: The Brouwer Fixed Point Theorem.

The degree is a topological obstruction to extend a map defined on the boundary to a map defined in the interior:

Proposition 1.7. *Suppose M is an m-dimensional oriented compact manifold with smooth and connected boundary ∂M, X a connected oriented $(m-1)$-manifold, and $f : \partial M \to X$ a smooth map that extends to a smooth map $g : M \to X$. Then $\deg(f) = 0$.*

Proof. Pick $\omega \in \Omega^{m-1}(X)$ so that $\int_X \omega = 1$. Then

$$\deg(f) = \deg(f)\int_X \omega = \int_{\partial M} f^*\omega = \int_{\partial M} \iota^* g^*\omega = \int_M d(g^*\omega) = \int_M g^* d\omega = 0,$$

where $\iota : \partial M \to M$ is the inclusion map, and in the last step we used the fact $d\omega = 0$. □

Corollary 1.8 (Brouwer Fixed Point Theorem)**.** *Every continuous map from B^m (=the unit ball in $\mathbb{R}^m$) to itself has a fixed point.*

Proof. Let $F_0 : B^m \to B^m$ be a continuous map without fixed point. Take a positive number

$$0 < r < \inf_{p \in B^n} |p - F_0(p)|/3.$$

Then by Whitney approximation theorem, there exists a smooth map $F_1 : B^m \to \mathbb{R}^m$ s.t.

$$|F_0(p) - F_1(p)| < r, \quad \forall p \in B^m.$$

Define $F : B^m \to B^m$ by $F(p) = F_1(p)/(1+r)$. Then $F : B^m \to B^m$ is a smooth map. F has no fixed point since

$$|F(p) - F_0(p)| \leq |F(p) - F_1(p)| + |F_1(p) - F_0(p)| < |F(p)|r + r \leq 2r < |F_0(p) - p|.$$

It follows that the smooth map

$$G : B^n \to S^{n-1}, \qquad p \mapsto \frac{p - F(p)}{|p - F(p)|}$$

is the extension of the smooth map

$$g = G|_{S^{n-1}} : S^{n-1} \to S^{n-1}.$$

By Proposition 1.7, $\deg(g) = 0$.

On the other hand, the map

$$H : S^{n-1} \times [0,1] \to S^{n-1}, \qquad (p,t) \mapsto \frac{p - tF(p)}{|p - tF(p)|}$$

is a homotopy between the identity map and g. So $\deg(g) = \deg(\mathrm{Id}) = 1$. Contradiction. $\square$

¶ More Applications.

Mapping degree has many other applications. For example,

- Given any two non-intersecting smooth curve $\gamma_i : S^1 \to \mathbb{R}^3 (i = 1, 2)$, we can define the *linking number* $\mathrm{Link}(\gamma_1, \gamma_2)$ to be

  $$\mathrm{Link}(\gamma_1, \gamma_2) := \deg(\Gamma_{\gamma_1,\gamma_2}),$$

  where $\Gamma_{\gamma_1,\gamma_2}$ is the Gauss map

  $$\Gamma_{\gamma_1,\gamma_2} : \mathbb{T}^2 \to S^2, \quad (e^{is}, e^{it}) \mapsto \frac{\gamma_1(e^{is}) - \gamma_2(e^{it})}{|\gamma_1(e^{is}) - \gamma_2(e^{it})|}.$$

  Geometrically, the linking number represents the number of times that each curve winds around the other, which may be positive or negative since we count the orientation of the two curves.
- One can also use the mapping degree to prove the fundamental theorem of algebra:

  $$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

  be any polynomial (with complex coefficients). We can regard f as a map S^2 from S^2. (how?) Then one can prove that the mapping degree of f is n (by showing that $f \sim f_0$ and computing $\deg(f_0)$, where $f_0(z) = z^n$.) So f admits at least one complex root. (Why?)
- We know that two properly homotopic maps have the same degree. Conversely degree can be used to characterize whether two maps are homotopic:

  Theorem 1.9 (Hopf degree theorem). *Let M be a compact connected oriented manifold of dimension m. Then two maps $f, g : M \to S^m$ are homotopic if and only if $\deg(f) = \deg(g)$.*
- One can prove that if $f : S^m \to S^m$ is an odd map, namely $f(-p) = -f(p)$ for all $p \in S^m$, then the degree of f is odd. This implies

  Theorem 1.10 (Borsuk-Ulam theorem). *For any continuous map $f : S^m \to \mathbb{R}^m$, there exists x_0 such that $f(-x_0) = f(x_0)$.*

2. The Poincaré duality and its applications

¶ The Poincaré duality.

Let M be an oriented manifold of dimension n. We have the following maps

- $\cup : H^k_{dR}(M) \times H^l_c(M) \to H^{k+l}_c(M), \quad ([\omega],[\eta]) \mapsto [\omega\wedge\eta]$.
- $\int_M : H^n_c(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$.

For any $0 \le k \le n$, consider the bilinear pairing map

$$P^k_M : H^k_{dR}(M) \times H^{n-k}_c(M) \to \mathbb{R}, \quad P^k_M([\omega],[\eta]) = \int_M \omega\wedge\eta.$$

The map P^k_M induces the following *Poincaré duality operator*

$$\mathcal{P}^k_M : H^k_{dR}(M) \to \left(H^{n-k}_c(M)\right)^*, \quad \mathcal{P}^k([\omega]) = \left\{\eta \mapsto \int_M \omega\wedge\eta\right\}.$$

For example, if M is connected, $\mathcal{P}^0_M$ maps the element $1 \in \mathbb{R} \simeq H^0_{dR}(M)$ to the linear map

$$\int_M : H^n_c(M) \to \mathbb{R}, \quad \eta \mapsto \int_M \eta$$

on $H^n_c(M)$, so that one can think of $\int_M$ as an element in $(H^n_c(M))^*$.

The major theorem we would like to discuss in this section is

Theorem 2.1 (Poincaré duality). *For any oriented manifold M and any k, the Poincaré duality map $\mathcal{P}^k_M$ is a linear isomorphism from $H^k_{dR}(M)$ to $\left(H^{n-k}_c(M)\right)^*$.*

Remark. If $\dim H^{m-k}_c(M) < \infty$, then $\left(H^{n-k}_c(M)\right)^*$ is isomorphic to $H^{n-k}_c(M)$. So we get

$$H^k_{dR}(M) \simeq H^{n-k}_c(M).$$

Example. For $M = \mathbb{R}^n$, we have

$$H^k_{dR}(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R}, & k=n, \\ 0, & k\neq n, \end{cases} \qquad \text{and} \qquad H^k_c(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R}, & k=0, \\ 0, & k\neq 0. \end{cases}$$

Example. For $M = S^n$, we have $H^k_c(S^n) = H^k_{dR}(S^n) \simeq \begin{cases} \mathbb{R}, & k=0,n, \\ 0, & k\neq 0,n. \end{cases}$

Example. For any connected oriented manifold of dimension n, we have already seen

$$H^0_{dR}(M) \simeq \mathbb{R} \simeq H^n_c(M)$$

and

$$H^n_{dR}(M) \simeq \left\{ \begin{array}{ll} \mathbb{R}, & M \text{ is compact} \\ 0, & M \text{ is non-compact} \end{array} \right\} \simeq H^0_c(M).$$

Example. Let $M = \cup_{i\in\mathbb{N}}(i,i+1)$ be a countable union of disjoint open intervals. Then

$$H^0_{dR}(M) \simeq \prod_{i\in\mathbb{N}} \mathbb{R} = \{(a_1,a_2,\cdots) \mid a_i \in \mathbb{R}\}$$

and (Note: $H^1_c((i,i+1)) \simeq \mathbb{R}$.)

$$H^1_c(M) \simeq \bigoplus_{i\in\mathbb{N}} \mathbb{R} = \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{R},\ \text{all but finitely many } a_i's \text{ are zero }\}.$$

A well-known fact (but non-trivial) in algebra:

$$\left(\bigoplus_{i\in\mathbb{N}} \mathbb{R}\right)^* = \prod_{i\in\mathbb{N}} \mathbb{R} \qquad \text{while} \qquad \left(\prod_{i\in\mathbb{N}} \mathbb{R}\right)^* \neq \bigoplus_{i\in\mathbb{N}} \mathbb{R}.$$

So we have Poincaré duality

$$H^k_{dR}(M) \simeq \left(H^{n-k}_c(M)\right)^*.$$

But in general

$$\left(H^k_{dR}(M)\right)^* \not\simeq H^{n-k}_c(M).$$

¶ The Poincaré duality: Sketch of a proof.

In what follows we will sketch a proof of Poincaré duality for oriented manifolds admitting a finite good cover, although the theorem holds for any oriented manifold. We need

Lemma 2.2. *The following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_{dR}(M) \arrow[r, "\alpha"] \arrow[d, "\mathcal{P}^k_M"'] & H^k_{dR}(U) \oplus H^k_{dR}(V) \arrow[r, "\beta"] \arrow[d, "\mathcal{P}^k_U \oplus \mathcal{P}^k_V"'] & H^k_{dR}(U \cap V) \arrow[r, "(-1)^{k+1}\delta"] \arrow[d, "\mathcal{P}^k_{U\cap V}"'] & H^{k+1}_{dR}(M) \arrow[r] \arrow[d, "\mathcal{P}^{k+1}_M"'] & \cdots \\
\cdots \arrow[r] & H^{n-k}_c(M)^* \arrow[r, "\alpha^*"'] & H^{n-k}_c(U)^* \oplus H^{n-k}_c(V)^* \arrow[r, "\beta^*"'] & H^{n-k}_c(U \cap V)^* \arrow[r, "\delta^*"'] & H^{n-k-1}_c(M)^* \arrow[r] & \cdots
\end{tikzcd}$$

where the bottom row is the dual of the Mayer-Vietoris sequence for compactly supported de Rham cohomology groups.

Sketch of proof of Poincaré duality for oriented manifolds M admitting a finite good cover. We proceed by induction. The theorem holds for M admitting one good chart, in which case $M \simeq \mathbb{R}^n$, and the isomorphism follows from the two versions of Poincaré lemma that we proved.

Now suppose the theorem holds for manifolds admitting a good cover of no more than $k-1$ open sets, and suppose M admits a good cover $\{U_1, \cdots, U_k\}$. We let

$$U = U_1 \cup \cdots \cup U_{k-1} \quad \text{and} \quad V = U_k.$$

Then U, V and $U \cap V$ all admit a good cover of no more than $k-1$ open sets. By induction hypothesis, $\mathcal{P}^k_U$, $\mathcal{P}^k_V$ and $\mathcal{P}^k_{U\cap V}$ are all isomorphisms. By the above lemma and the Five Lemma (see Lecture 26), $\mathcal{P}^k_M$ is an isomorphism. $\square$

¶ Application 1: The Künneth formula for $H_c^k(M \times N)$.

As a corollary, we get (the result actually holds for any manifold M)

Corollary 2.3. *For any oriented manifold M whose compact supported cohomology groups are finite dimensional,*

$$H_c^{k+l}(M \times \mathbb{R}^l) \simeq H_c^k(M).$$

Proof. By Poincaré duality and homotopy invariance of the de Rham cohomology groups,

$$H_c^{k+l}(M \times \mathbb{R}^l) \simeq H_{dR}^{n+l-k-l}(M \times \mathbb{R}^l) \simeq H_{dR}^{n-k}(M) \simeq H_c^k(M).$$

☐

More generally, one can prove (the result actually holds for any manifold M)

Corollary 2.4 (Künneth formula). *If M, N are orientable with finite good cover, then*

$$H_c^k(M \times N) \simeq \bigoplus_{i=0}^{k} H_c^i(M) \otimes H_c^{k-i}(N).$$

Proof. By Poincaré Duality and the Künneth formula for the de Rham cohomology groups,

$$H_c^k(M \times N) \simeq H_{dR}^{m+n-k}(M \times N) \simeq \bigoplus_{i=0}^{m+n-k} H_{dR}^i(M) \otimes H_{dR}^{m+n-k-i}(N).$$

The indices i should satisfy $i \le m$ and $m+n-k-i \le n$, i.e. $m-k \le i \le m$. Thus

$$H_c^k(M \times N) \simeq \bigoplus_{i=m-k}^{m} H_{dR}^i(M) \otimes H_{dR}^{m+n-k-i}(N) \simeq \bigoplus_{i=0}^{k} H_{dR}^{m-i}(M) \otimes H_{dR}^{n-k+i}(N).$$

Now the result follows from the Poincaré duality. ☐

¶ Application 2: The Betti numbers and the Euler characteristics.

We also give a couple applications to the Betti numbers and the Euler characteristics. Recall that the Betti number

$$b_k = \dim H_{dR}^k(M),$$

and the Euler characteristic

$$\chi(M) = \sum_k (-1)^k b_k.$$

We have

Proposition 2.5. *Let M be a compact orientable manifold of dimension m, then*

1. *For any k, $b_k = b_{m-k}$.*
2. *If $m = 4n+2$, then b_{2n+1} is even.*

Proof. (1) This follows from the fact

$$H^k_{dR}(M) \simeq H^{n-k}_c(M) = H^{n-k}_{dR}(M).$$

(2) follows from the non-degeneracy of the pairing

$$P^{2n+1}_M : H^{2n+1}_{dR}(M) \times H^{2n+1}_{dR}(M) \to \mathbb{R}.$$

If fact, for any $[\omega], [\eta] \in H^{2n+1}_{dR}(M)$, we have

$$P^{2n+1}_M([\omega],[\eta]) = \int_M \omega \wedge \eta = \int_M (-1)^{(2n+1)(2n+1)} \eta \wedge \omega = -P^{2n+1}_M([\eta],[\omega]).$$

It follows that the matrix for the pairing $P^{2n+1}_M : H^{2n+1}_{dR}(M) \times H^{2n+1}_{dR}(M) \to \mathbb{R}$ is an anti-symmetric $b_{2n+1} \times b_{2n+1}$ matrix. So

$$\det(P^{2n+1}_M) = \det((P^{2n+1}_M)^T) = (-1)^{b_{2n+1}} \det(P^{2n+1}_M).$$

So b_{2n+1} must be an even number, otherwise $\det(P^{2n+1}_M) = 0$ and thus P^{2n+1}_M is not a non-degenerate pairing. □

As consequences, we get Finally we prove

Theorem 2.6. *Let M be a compact oriented manifold.*

1. *If $\dim M = 2n+1$, then $\chi(M) = 0$.*
2. *If $\dim M = 4n+2$, then $\chi(M)$ is even.*

Proof. (1) If $\dim M = 2n+1$, we have

$$\begin{aligned}\chi(M) = \sum_{k=0}^{2n+1} (-1)^k b_k &= \sum_{k=0}^{n} (-1)^k b_k + \sum_{k=n+1}^{2n+1} (-1)^k b_{2n+1-k} \\ &= \sum_{k=0}^{n} ((-1)^k + (-1)^{2n+1-k}) b_k \\ &= 0.\end{aligned}$$

(2) Suppose $\dim M = 4n+2$, then the same argument yields

$$\chi(M) = \sum_{k=0}^{4n+2} (-1)^k b_k = \sum_{k=0}^{2n} ((-1)^k + (-1)^{4n+2-k}) b_k + b_{2n+1}.$$

Since $(-1)^k + (-1)^{4n+2-k} = \pm 2$ and since b_{2n+1} is even, the result follows. □

¶ The Poincaré dual of a submanifold.

Now let M be an oriented manifold of dimension m and let $\iota : S \hookrightarrow M$ be a closed oriented submanifold of dimension $k \leq m$. Then the map (why it is well-defined?)

$$\int_S : H_c^k(M) \to \mathbb{R}, \quad [\eta] \mapsto \int_S \iota^* \eta$$

defines an element $\int_S$ in $\left(H_c^k(M)\right)^*$, and thus by Poincaré duality, it defines an element $\mathrm{Pd}_M(S)$ in $H_{dR}^{m-k}(M)$, characterized by the following property: if $\omega \in Z^{m-k}(M)$ is a representative of $\mathrm{Pd}_M(S) \in H_{dR}^{m-k}(M)$, then for any $[\eta] \in H_c^k(M)$, we have

$$\int_S \iota^* \eta = \int_M \eta \wedge \omega.$$

Definition 2.7. We call $\mathrm{Pd}_M(S)$ the *Poincaré dual* of S in M.

Example. Let M be closed and oriented. Then M itself is a closed oriented submanifold of M. By definition, the element in $H_{dR}^0(M)$ corresponding to the (sub-)manifold M is $[1]$, i.e.

$$\mathrm{Pd}_M(M) = [1].$$

Example. Let M be a closed oriented manifold of dimension m, and let

$$\Delta = \{(x, x) \mid x \in M\} \subset M \times M$$

be the diagonal submanifold. We would like to calculate $\mathrm{Pd}_{M\times M}(\Delta) \in H_{dR}^m(M \times M)$. We denote by $pr_1 : M \times M \to M$ and $pr_2 : M \times M \to M$ the two canonical projections of $M \times M$ to the two components. Let

$$\{[\omega_i^j] \mid 1 \leq i \leq b_j\}$$

be a basis of $H_{dR}^j(M)$, where $b_j = \dim H_{dR}^j(M)$ is the j, and let

$$\{[\nu_i^{m-j}] \mid 1 \leq i \leq b_{m-j} = b_j\}$$

be the dual basis of $H_{dR}^{m-j}(M)$ under Poincaré duality, namely,

$$\int_M \omega_i^j \wedge \nu_k^{m-j} = \delta_{ik}, \qquad \forall 1 \leq i, k \leq b_j.$$

Then we have

$$\begin{aligned}\int_{M\times M} (pr_1^*\omega_i^j \wedge pr_2^*\nu_k^{m-j}) \wedge (pr_1^*\nu_s^{m-r} \wedge pr_2^*\omega_t^r) &= (-1)^{m(m-j)} \int_{M\times M} pr_1^*(\omega_i^j \wedge \nu_s^{m-r}) \wedge pr_2^*(\omega_t^r \wedge \nu_k^{m-j}) \\ &= (-1)^{m(m-j)} \delta_{jr} \int_M (\omega_i^j \wedge \nu_s^{m-j}) \int_M (\omega_t^j \wedge \nu_k^{m-j}) \\ &= (-1)^{m(m-j)} \delta_{rj} \delta_{is} \delta_{tk},\end{aligned}$$

which implies that

$$\{[pr_1^*\omega_i^j \wedge pr_2^*\nu_k^{m-j}] \mid 0 \leq j \leq m, 1 \leq i, k \leq b_j\}$$

are linearly independent. By Künneth formula and dimension counting, they form a basis of $H^m_{dR}(M\times M)$. Thus we can write

$$\mathrm{Pd}_{M\times M}(\Delta)=\sum_{i,j,k} c_j^{i,k}[pr_1^*\omega_i^j\wedge pr_2^*\nu_k^{m-j}].$$

To calculate the coefficients, we let $\omega\in Z^m(M\times M)$ be a representative of $\mathrm{Pd}_{M\times M}(\Delta)$ and for each r,s,t, calculate both sides of

$$\int_{M\times M}(pr_1^*\nu_s^{m-r}\wedge pr_2^*\omega_t^r)\wedge\omega=\int_\Delta \iota^*(pr_1^*\nu_s^{m-r}\wedge pr_2^*\omega_t^r).$$

Since $[\omega]=\mathrm{Pd}_{M\times M}(\Delta)$, the left hand side gives

$$\begin{aligned}LHS&=(-1)^m\sum_{i,j,k}c_j^{i,k}\int_{M\times M}(pr_1^*\omega_i^j\wedge pr_2^*\nu_k^{m-j})\wedge(pr_1^*\nu_s^{m-r}\wedge pr_2^*\omega_t^r)\\&=(-1)^m\sum_{i,j,k}c_j^{i,k}(-1)^{m(m-j)}\delta_{rj}\delta_{is}\delta_{tk}\\&=(-1)^{mr}c_r^{s,t}.\end{aligned}$$

For the right hand side, if we denote $\varphi:M\to\Delta$ be the orientation-preserving (why) diffeomorphism given by $\varphi(x)=(x,x)$, then $pr_1\circ\iota\circ\varphi=Id=pr_1\circ\iota\circ\varphi$ and thus

$$RHS=\int_M\varphi^*\iota^*(pr_1^*\nu_s^{m-r}\wedge pr_2^*\omega_t^r)=\int_M\nu_s^{m-r}\wedge\omega_t^r=(-1)^{r(m-r)}\delta_{st}.$$

So we conclude $c_r^{s,t}=(-1)^{r^2}\delta_{st}=(-1)^r\delta_{st}$, and thus

$$\mathrm{Pd}_{M\times M}(\Delta)=\sum_{i,j}(-1)^j[pr_1^*\omega_i^j\wedge pr_2^*\nu_i^{m-j}].$$

¶ The Lefschetz fixed point theorem.

More generally, let $f:M\to M$ be a smooth map, and

$$\Gamma_f=\{(x,f(x))\mid x\in M\}\subset M\times M$$

be its graph. Then the pull-back $f^*:H^*_{dR}(M)\to H^*_{dR}(M)$ is linear and thus has the form

$$f^*(\omega_i^j)=\sum_k A_j^{ik}\omega_k^j.$$

By the same computation as above, one can show (exercise)

$$\mathrm{Pd}_{M\times M}(\Gamma_f)=\sum_{i,j,k}(-1)^jA^j_{ik}[pr_1^*\omega_i^j\wedge pr_2^*\nu_k^{m-j}].$$

As a consequence, we have

Proposition 2.8. *Let $\iota:\Delta\to M\times M$ be the inclusion map. Then*

$$\int_\Delta\iota^*\mathrm{Pd}_{M\times M}(\Gamma_f)=\sum_j(-1)^j\mathrm{tr}(f^*|_{H^j_{dR}(M)}).$$

Proof. Let $\varphi : M \to \Delta$ be given by $\varphi(x) = (x, x)$ as above. Then

$$\begin{aligned}\int_\Delta \iota^* \mathrm{Pd}_{M\times M}(\Delta) = \int_M \varphi^* \iota^* \mathrm{Pd}_{M\times M}(\Delta) &= \sum_j (-1)^j \sum_{i,k} A^j_{ik} \int_M \omega^j_i \wedge \nu^{m-j}_k \\ &= \sum_j (-1)^j \sum_{i,k} A^j_{ii} \\ &= \sum_j (-1)^j \mathrm{tr}(f^*|_{H^j_{dR}(M)}). \end{aligned}$$ □

This result has a very important application. Recall from PSet 4-1-4:

> Let $f : M \to M$ be a smooth map. A point $p \in M$ is a called a *fixed point* of f if $f(p) = p$. We say f is a *Lefschetz map* if for each fixed point p of f, 1 is not an eigenvalue of $df_p : T_pM \to T_pM$. The *local Lefschetz number* $L_p(f)$ of a Lefschetz map at a fixed point p is the sign of the determinant $\det(df_p - \mathrm{Id})$, i.e. $L_p(f) := 1$ if $\det(df_p - \mathrm{Id}) > 0$, and $L_p(f) := -1$ if $\det(df_p - \mathrm{Id}) < 0$.

Theorem 2.9 (Lefschetz fixed point theorem)**.** *Let $f : M \to M$ be a Lefschetz map. Then*

$$\sum_{p\in Fix(f)} L_p(f) = \sum_j (-1)^j \mathrm{tr}(f^*|_{H^j_{dR}(M)}).$$

I will not give a detailed proof since it was assigned as an "A^+-project".

¶ The Poincaré-Hopf theorem.

Finally we let M be a compact oriented smooth manifold, and X a smooth vector field on M. Then for t small enough, the fixed points of the flow $\varphi^t : M \to M$ generated by X are exactly the zeroes of X. Let x be a zero of X, then we get a family of linear maps $d\varphi^t_x : T_xM \to T_xM$, and thus a linear map

$$A_x := \left.\frac{d}{dt}\right|_{t=0} (d\varphi^t_x).$$

Definition 2.10. We say x is a *nondegenerate zero* of X if $\det A_x \neq 0$. In this case we call the number

$$\mathrm{Ind}(X, x) = \mathrm{sgn}(\det A_x)$$

the *index* of X at x.

Remark. Let x_0 be a non-degenerate zero of a smooth vector field X on M. Choose a local coordinate system near x_0 so that x_0 is the only zero of X. Then X defines a vector field $\widetilde{X}$ on $V \subset \mathbb{R}^n$ with a non-degenerate zero at 0. In particular, for r small enough, $\widetilde{X}$ has no zero on the sphere $S(r) = \{x \in V \mid |x| = r\}$. Consider the map $F_r : S(r) \to S^{n-1}$ by

$$F_r(x) = \widetilde{X}(x)/|\widetilde{X}(x)|.$$

Then one can prove: $\deg F_r = \mathrm{Ind}(X, x_0)$. In other words, the index of a vector field at each non-degenerate critical point is a mapping degree.

Note that by definition

$$\det(d\varphi^t_x - I) = \det(tA_x + h.o.t.) = t^m \det A_x + h.o.t.$$

and thus $L_x(\varphi^t) = \mathrm{Ind}(X, x)$. So we get

$$\sum_j \mathrm{Ind}(X, x_j) = \sum_{x\in Fix(\varphi^t)} L_x(\varphi^t) = \sum_j (-1)^j \mathrm{tr}((\varphi^t)^*|_{H^j_{dR}(M)}).$$

But φ^t is homotopic to the identity map, and thus $(\varphi^t)^* = Id$, and thus we proved the following famous theorem that I promised to some of you last semester:

Theorem 2.11 (Poincaré-Hopf theorem). *Let M be a compact oriented smooth manifold, and X a smooth vector field on M, with only non-degenerate zeroes $x_1, \cdots, x_k$. Then*

$$\sum_j \mathrm{Ind}(X, x_j) = \chi(M).$$

3. READING: GAUSS-BONNET-CHERN THEOREM

¶ Vector bundles.

We need some definitions:

Definition 3.1. Let E, M be smooth manifolds, and $\pi : E \to M$ a surjective smooth map. We say (π, E, M) is a *vector bundle of rank* r if for every $p \in M$, there exits an open neighborhood U_α of p and a diffeomorphism (called the *local trivialization*)

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

so that

1. $E_p = \pi^{-1}(p)$ is a r dimensional vector space, and $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^r$ is a linear map.
2. For $U_\alpha \cap U_\beta \neq \emptyset$, there is a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ so that
$$\Phi_\beta \circ \Phi_\alpha^{-1}(m, v) = (m, g_{\beta\alpha}(m)(v)), \quad \forall m \in U_\alpha \cap U_\beta, v \in \mathbb{R}^r.$$

We will call E the *total space*, M the *base* and $\pi^{-1}(p)$ the *fiber* over p. (In the case there is no ambiguity about the base, we will denote a vector bundle by E for short.)

For example, we have seen the following vector bundles:

- the tangent bundle TM of a smooth manifold M.
- the cotangent bundle T^*M of a smooth manifold M.
- the normal bundle N_SM of a submanifold S in M

One may define the conception of *vector bundle isomorphism* (try to define this by yourself), and prove (exercise)

Lemma 3.2. *Let $\Delta \subset M \times M$ be the diagonal submanifold. Then $N\Delta$ is isomorphic to $T\Delta$.*

Roughly speaking, a vector bundle E over M is "a smooth varying family of vector spaces parameterized by a base manifold M". It is not hard to get

Proposition 3.3. *For any vector bundle E over M, one has*

$$H^k_{dR}(E) = H^k_{dR}(M), \qquad \forall k.$$

Proof. This is a consequence of the homotopy invariance: E is homotopy equivalent to M, since if we let $s_0 : M \to E$ be the zero section, then $\pi \circ s_0 = \mathrm{Id}_M$, and $s_0 \circ \pi \sim \mathrm{Id}_E$ via the homotopy

$$F : E \times \mathbb{R} \to E, \qquad (x, v, t) \mapsto (x, tv). \qquad \square$$

Again by Poincaré duality, we get the following generalization of Corollary 2.3:

Theorem 3.4 (Thom isomorphism)**.** *Let E be a rank r vector bundle over M. Assume both E and M are oriented with finite dimensional compact supported de Rham cohomology groups, then*

$$H^k_c(E) \simeq H^{k-r}_c(M), \qquad \forall k.$$

Proof. We apply Poincaré duality twice:

$$H^k_c(E) \simeq H^{n+r-k}_{dR}(E) \simeq H^{n+r-k}_{dR}(M) \simeq H^{k-r}_c(M). \qquad \square$$

¶ The Thom class and the Euler class.

Now let M be an oriented, compact and connected smooth n-manifold, and E an oriented vector bundle of rank r over M. Then the zero section

$$s_0 : M \to E, x \mapsto (x, 0)$$

defines an embedding of M into E.

Definition 3.5. We call $\tau(E) := \mathrm{Pd}_E(M) \in H^r_c(E)$ the *Todd class* of (π, E, M).

Remark. The constant function 1 on M gives us a degree-0 cohomology class $[1] \in H^0_{dR}(M)$. So the Thom isomorphism above produces an element

$$\tau(E) \in H^r_c(E).$$

One can show that this is the same as the Poincaré dual of M in E.

Let E be an oriented vector bundle of rank r over an oriented connected compact smooth manifold M. Let $s : M \to E$ be any global section of E, that is, a smooth map such that

$$\pi \circ s = \mathrm{Id}_M.$$

Then by using the pull-back, we get an element

$$s^*(\tau(E)) \in H^r_{dR}(M).$$

Proposition 3.6. *The de Rham cohomology class $s^*(\tau(E))$ is independent of the choices of s.*

Proof. Let $s_0 : M \to E$ be the zero section. Then $s_0 \sim s$ via the homotopy

$$F : M \times \mathbb{R}, (m, t) \mapsto (m, ts(m)).$$

So $s^*(\tau(E)) = s_0^*(\tau(E))$. $\square$

Definition 3.7. The de Rham cohomology class

$$e(E) := s^*(\tau(E)) \in H^r_{dR}(M)$$

is called the *Euler class* of E.

¶ Gauss-Bonnet-Chern theorem.

The following theorem is an important part of Gauss-Bonnet-Chern theorem:

Theorem 3.8. *Let M be a compact oriented manifold Then*

$$\int_M e(TM) = \chi(M).$$

Proof. Since Δ is diffeomorphic to M, and since $T\Delta \simeq N\Delta$ as vector bundles over Δ,

$$\int_M e(TM) = \int_\Delta e(T\Delta) = \int_\Delta e(N\Delta).$$

But by definition and the characterization formula of Poincaré dual,

$$\int_\Delta e(N\Delta) = \int_\Delta s_0^*\tau(N\Delta) = \int_\Delta s_0^* PD_{N\Delta}(\Delta) = \int_{N\Delta} PD_{N\Delta}(\Delta) \wedge PD_{N\Delta}(\Delta).$$

By tubular neighborhood theorem, we get

$$\int_{N\Delta} PD_{N\Delta}(\Delta) \wedge PD_{N\Delta}(\Delta) = \int_{M\times M} PD_{M\times M}(\Delta) \wedge PD_{M\times M}(\Delta) = \int_\Delta \iota^*\mathrm{PD}_{M\times M}(\Delta),$$

where $\iota : \Delta \to M \times M$ is the canonical embedding. But if we apply Proposition 2.8 to the identity map, we have

$$\int_\Delta \iota^*\mathrm{Pd}_{M\times M}(\Delta) = \chi(M).$$

So we get the desired formula. □

There is a geometric way to define the Euler class: One first fix a connection ∇ on E, and let R_∇ be the curvature 2-form of ∇. Then according to the famous Chern-Weil theory, if the rank of E is even, then $\chi(E)$ can be calculated via R_∇,

$$\chi(E) = [Pf(R_\nabla)]$$

where Pf is defined as follows: For any $A = (a^i_j) \in \mathfrak{so}(r)$,

$$\mathrm{Pf}(A) = \frac{1}{(4\pi)^{r/2}(r/2)!}\sum_{\sigma\in S_r}(-1)^\sigma a^{\sigma(1)}_{\sigma(2)} a^{\sigma(3)}_{\sigma(4)} \cdots a^{\sigma(r-1)}_{\sigma(r)}.$$

It is homogeneous polynomial of degree $r/2$.

Finally we assume M is a closed oriented smooth manifold of dimension $m = 2n$. Endow with M a Riemannian metric, and consider the tangent bundle $E = TM$. Then we have a Levi-Civita connection ∇ on TM, and we arrive at the famous

Theorem 3.9 (Gauss-Bonnet-Chern). *We have*

$$\int_M Pf(R_\nabla) = \chi(M).$$

Dedicated to
Shiing-Shen Chern (1911-2004)
for his 110th Birth Anniversary