# PROBLEM SET 1, PART 1: SMOOTH MANIFOLDS
# DUE: SEP. 28

(1) [**Topological group**]
A topological group is a topological space $G$ which is also a group, so that the multiplication operation

$$\mu : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2$$

and the inverse operation

$$i : G \to G, \quad g \mapsto g^{-1}$$

are both continuous. For any subsets $S$ and $T$ in $G$ we can define

$$ST = \{g_1 g_2 \mid g_1 \in S, g_2 \in T\}.$$

By this way we can define the subsets $S^n = S \times S \times \cdots \times S$ and $S^{-1}$.

(a) If $G$ is a topological group, and $U$ is any open neighborhood of the identity element $e \in G$. Prove: There exists an open neighborhood $V$ of $e$ so that $V = V^{-1}$ and $V^2 \subset U$.

(b) Prove: If $G$ is a connected topological group, then for any open neighborhood $U$ of the identity element $e \in G$, we have

$$G = \bigcup_{n=1}^{\infty} U^n.$$

(c) [Not required] Suppose $G$ is compact Hausdorff topological group and $g \in G$. Prove: $e \in \overline{\{g^n \mid n \in \mathbb{Z} \setminus \{0\}\}}$.

(2) [**Locally Euclidean**]
Prove the following properties of locally Euclidean spaces:

(a) Any connected component of a locally Euclidean space is open.

(b) Any connected locally Euclidean space is path connected.

(c) Any locally Euclidean Hausdorff space is regular. [Thus as a consequence of Urysohn's metrization theorem, any topological manifold is metrizable.]

(d) [Not required] If both $X, Y$ are connected, second countable and locally Euclidean, and $f : X \to Y$ is bijective and continuous, then $f$ is a homeomorphism.

(3) [**Topological manifolds with boundary**]

(a) Find the definition of topological manifolds with boundary from literature.

(b) Prove: If $M$ is a topological $n$-manifold with boundary, then its boundary, $\partial M$, is a topological $(n-1)$-manifold without boundary.

(c) [Not required] Prove: the product of two topological manifolds with boundaries is a topological manifold with boundary. What is its boundary?

(4) [**Construct smooth manifolds by gluing Euclidean open sets**]
Let $M$ be a smooth manifold with atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha, V_\alpha)\}$.

(a) Prove: the transition maps $\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfy the *cocycle conditions*
   (i) $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ on $\varphi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma)$.
   (ii) $\varphi_{\alpha\alpha} = \mathrm{Id}_{V_\alpha}$.
   (iii) $\varphi_{\alpha\beta} = (\varphi_{\beta\alpha})^{-1}$.

(b) Now on the <u>disjoint union</u> $\widetilde{M} := \bigsqcup_\alpha V_\alpha$ we define an equivalence relation via

$$x \sim y \Longleftrightarrow \exists \alpha, \beta \text{ s.t. } x \in V_\alpha, y \in V_\beta \text{ and } y = \varphi_{\alpha\beta}(x).$$

   (i) Check: $\sim$ is an equivalence relation on $\widetilde{M}$.
   (ii) Prove: the quotient $\widetilde{M}/\sim$ is homeomorphic to $M$.
   (iii) Define a natural smooth structure on $\widetilde{M}/\sim$

(5) [**Complex manifolds**]

(a) Write down the definition of complex manifolds.
(b) Define $\mathbb{CP}^n$ and prove that it is a complex manifold.
(c) Slightly adjust the charts constructed by stereographic projections in class to prove that $S^2$ is a complex manifold. Does the same argument apply to $S^{2n}$?

(6) [**Product of manifolds**]

(a) Prove: The product of two topological manifolds is a topological manifold.
(b) Let $M$, $N$ be smooth manifolds with atlases $\{(\phi_\alpha, U_\alpha, V_\alpha)\}$ and $\{(\psi_\beta, X_\beta, Y_\beta)\}$ respectively. Define an atlas on the product space $M \times N$ to be $\{(\phi_\alpha \times \psi_\beta, U_\alpha \times X_\beta, V_\alpha \times Y_\beta)\}$. Check that this gives a smooth structure on $M \times N$.
(c) Explicitly construct local charts on $\mathbb{T}^2 = S^1 \times S^1 \subset \mathbb{R}^4$ to make it into a smooth manifold. (You need to prove that your charts are compatible.)