Topology (H) Lecture 3
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TOPOLOGY: DEFINITIONS AND EXAMPLES

1. Continuous maps between metric spaces: continued

¶ More metrics that induce equivalent conceptions of continuity.

Let's study one more example.

Example 1.1. Consider another pair of metrics on $\mathbb{R}^n$, say the Euclidian metric $d_2(x, y) = |x - y|$ and the bounded metric

$$\bar{d}_2(x, y) := \frac{d_2(x, y)}{1 + d_2(x, y)}$$

induced by d_2. Obviously

$$\bar{d}_2(x, y) \leq d_2(x, y).$$

But d_2 and $\bar{d}_2$ are **not strongly equivalent**, since given any constant $C > 0$, there exists $x, y \in \mathbb{R}^n$ such that

$$d_2(x, y) > C \geq C\bar{d}_2(x, y).$$

However, if we study the conception of continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$, again we will arrive at the same conclusion: a function $f : (\mathbb{R}^n, d_2) \to \mathbb{R}$ is continuous if and only if the function $f : (\mathbb{R}^n, \bar{d}_2) \to \mathbb{R}$ is continuous:

In fact, if $f : (\mathbb{R}^n, \bar{d}_2) \to \mathbb{R}$ is continuous, then $f : (\mathbb{R}^n, d_2) \to \mathbb{R}$ is continuous since $\bar{d}_2(x, y) \leq d_2(x, y)$.

Conversely if $f : (\mathbb{R}^n, d_2) \to \mathbb{R}$ is continuous, namely,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, d_2(x, x_0) < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

Since

$$\bar{d}_2(x, y) < \frac{\delta}{1 + \delta} \Longrightarrow \frac{d_2(x, y)}{1 + d_2(x, y)} < \frac{\delta}{1 + \delta} \Longrightarrow d_2(x, y) < \delta,$$

we find that

$$\forall \varepsilon > 0, \exists \delta' = \frac{\delta}{1 + \delta} > 0 \text{ s.t. } \forall x \in X, \bar{d}_2(x, x_0) < \delta' \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

In other words, $f : (\mathbb{R}^n, \bar{d}_2) \to \mathbb{R}$ is continuous.

From the above examples one can imagine that there should be some underlying structure that is more fundamental than the metric structure that induces the conception of continuity.

¶ Local continuity via neighborhoods.

To figure out the structure behind continuity, let's recall

$$\begin{aligned} &\text{A map } f:(X,d_X)\to(Y,d_Y) \text{ is continuous at } x_0\in X\\ \iff &\forall\varepsilon>0,\exists\delta>0 \text{ s.t. } \forall x\in X, d_X(x,x_0)<\delta, \text{ we have } d_Y(f(x),f(x_0))<\varepsilon\\ \iff &\forall\varepsilon>0,\exists\delta>0 \text{ s.t. } B^X(x_0,\delta)\subset f^{-1}\big(B^Y(f(x_0),\varepsilon)\big).\end{aligned}$$

Of course these equivalent characterizations all depends on the metric structure (either in terms of the metric d, or in terms of the metric balls). To get rid of the "metric dependence", let's recall that a subset $U\subset X$ is *open* if

$$\forall x\in U,\exists\varepsilon=\varepsilon(x)>0 \text{ such that } B(x,\varepsilon)\subset U.$$

Intuitively, continuity of f at a point x_0 concerns only points in X near x_0 and points in Y near $f(x)$. Using open sets, we can introduce the following definition of neighborhoods in which the metric does not appear explicitly:

Definition 1.2. We say a subset $N\subset X$ is a *neighborhood* of x if there exists an open set U in X so that $x\in U\subset N$.[1]

Remark 1.3. If we denote by $\mathscr{N}(x)$ the set of all neighborhoods of x, it is easy to see

(1) If $N\in\mathscr{N}(x)$, then $x\in N$.
(2) If $M\supset N$ and $N\in\mathscr{N}(x)$ then $M\in\mathscr{N}(x)$.
(3) If $N_1,N_2\in\mathscr{N}(x)$, then $N_1\cap N_2\in\mathscr{N}(x)$.
(4) If $N\in\mathscr{N}(x)$, then $\exists M\subset N$ and $M\in\mathscr{N}(x)$, s.t. $\forall y\in M, N\in\mathscr{N}(y)$.

It turns out that we can characterize continuity at a point via neighborhoods:

Proposition 1.4. *Let $f:(X,d_X)\to(Y,d_Y)$ be a map between metric spaces. Then f is continuous at $x\in X$ if and only if the pre-image of any neighborhood of $f(x)$ is a neighborhood of x.*

Proof. Suppose f is continuous at $x\in X$, and $M\subset Y$ is a neighborhood of x. Then by definition, there is an open set $V\subset Y$ such that $f(x)\in V\subset M$. By definition of open set, $\exists\varepsilon>0$ s.t. $B(f(x),\varepsilon)\subset V$. By continuity of f at x, $\exists\delta>0$ such that

$$B(x,\delta)\subset f^{-1}\left(B(f(x),\varepsilon)\right)\subset f^{-1}(V)\subset f^{-1}(M).$$

So $f^{-1}(M)$ is a neighborhood of x.

Conversely suppose for any neighborhood $M\subset Y$ of $f(x)$, $f^{-1}(M)$ is a neighborhood of x. Then in particular for $\forall\varepsilon>0, f^{-1}\left(B(f(x),\varepsilon)\right)$ is a neighborhood of x, i.e. it contains an open set U with $x\in U$. By the definition of open set, $\exists\delta>0$ s.t. $B(x,\delta)\subset U$, which implies $B(x,\delta)\subset f^{-1}\left(B(f(x),\varepsilon)\right)$. So f is continuous at x. ☐

[1] In some books (including Munkres' book) people require neighborhoods to be open. We will not make such requirement. Instead, we use the expression "an open neighborhood of x" to indicate a set which is both open and is a neighborhood of x.

Remark 1.5. In general, if f is continuous at x_0, it may happen that the pre-image of on open neighborhood of $f(x_0)$ is not open in X. [Try to find an example!]

¶ Global continuity via open sets.

As a consequence of Proposition 1.4, we get the following characterization of (globally) continuous maps between abstract metric spaces:

Theorem 1.6. *A map $f : (X, d_X) \to (Y, d_Y)$ is continuous if and only if for any open set V in Y, the pre-image $f^{-1}(V)$ is open in X.*

Proof. Suppose f is continuous, and $V \subset Y$ is open. Then $\forall x \in f^{-1}(V)$, by Proposition 1.4, $f^{-1}(V)$ contains an open set U with $x \in U$. So $f^{-1}(V)$ is open in X.

Conversely suppose for any open set $V \subset Y$, the pre-image $f^{-1}(V)$ is open in X. For any $x \in X$, take any open set V in Y with $f(x) \in V$. Then $f^{-1}(V)$ itself is an open set in X which contains the point x. So by Proposition 1.4, f is continuous. □

Definition 1.7 (Topologically equivalent metrics)**.** Let d_1 and d_2 be two metrics on a set X. We say d_1 and d_2 are *topologically equivalent* if they produce the same set of open sets.

Obviously strongly equivalent metrics are always topologically equivalent, but the converse is not true. In general, we will call a conception a "topological conception" if the conception depends only on the collection of open sets (this will be clear later). So "neighborhood" is a topological conception, i.e. it depends only on the collection of open sets, and continuity is a topological property.

As a consequence of Theorem 1.6, we have

Corollary 1.8. *Suppose $\widetilde{d_X}$ and $\widetilde{d_Y}$ are metrics that are topologically equivalent to d_X and d_Y respectively, then $f : (X, d_X) \to (Y, d_Y)$ is continuous if and only if $f : (X, \widetilde{d_X}) \to (Y, \widetilde{d_Y})$ is continuous.*

This is why $d_1, d_2, \bar{d}_2$ on $\mathbb{R}^n$ produce the same set of continuous functions, while the discrete metric produce a different set: it is not hard to see from the arguments in the examples above that the collection of open sets determined by d_1, d_2 or $\bar{d}_2$ are all the same, while the collection of open sets determined by the discrete metric is different!

In conclusion:

> Although we defined continuity via the metric structure, continuity is really a conception that depends only on the collection of open sets produced by the metric!

¶ A non-topological conception: the uniform continuity.

To compare, let's take a look at a similar conception: the "uniform continuity" of a map between metric spaces. The definition is straightforward:

Definition 1.9. A map $f:(X,d_X)\to(Y,d_Y)$ is *uniformly continuous* if

$$\forall\varepsilon>0,\exists\delta>0,\text{ s.t. } d_X(x_1,x_2)<\delta\Rightarrow d_Y(f(x_1),f(x_2))<\varepsilon.$$

Of course uniformly continuous functions are continuous, but the converse is not true. It turns out that "uniform continuity" is NOT a topological property: it does depend on the metric.[2]

Example 1.10. Let d be the standard metric on $\mathbb{R}$, and let d_1 be the metric on $\mathbb{R}$ induced by the map $\arctan:\mathbb{R}\to(-\frac{\pi}{2},\frac{\pi}{2})$, i.e.

$$d_1(x,y):=|\arctan(x)-\arctan(y)|.$$

Then open balls of d_1 are exactly open intervals in $\mathbb{R}$. So d and d_1 induces the same set of open sets, i.e. they are topologically equivalent.

Consider the identity map

$$f:\mathbb{R}\to\mathbb{R}, f(x)=x.$$

Then $f:(\mathbb{R},d)\to(\mathbb{R},d)$ is uniformly continuous, but $f:(\mathbb{R},d_1)\to(\mathbb{R},d)$ is NOT uniformly continuous since

$$d_1(n,n+1)=|\arctan(n)-\arctan(n+1)|\to 0 \text{ as } n\to\infty$$

but $d(n,n+1)=1$.

2. Topology: Definitions and Examples

¶ Topology via neighborhood structure.

To extend the conceptions of continuity and convergence to more general "spaces", intuitively one need to axiomatize the conception of "neighborhood" first. Here is how to do this:

For any $x\in X$, one can assign to it a non-empty collection of subsets,

$$x\mapsto\mathscr{N}(x)\subset\mathcal{P}(X),^{3}$$

with the understanding that each element in $\mathscr{N}(x)$ is a neighborhood of x. The axioms for these $\mathscr{N}(x)$'s to satisfy are the following:

(N1) If $N\in\mathscr{N}(x)$, then $x\in N$.
(N2) If $M\supset N$ and $N\in\mathscr{N}(x)$ then $M\in\mathscr{N}(x)$.
(N3) If $N_1,N_2\in\mathscr{N}(x)$, then $N_1\cap N_2\in\mathscr{N}(x)$.
(N4) If $N\in\mathscr{N}(x)$, then $\exists M\subset N$ and $M\in\mathscr{N}(x)$, s.t. $\forall y\in M, N\in\mathscr{N}(y)$.

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[2] There is a generalization of metric structure, called the "uniform structure". One can define uniform continuity for maps between spaces with uniform structures. For details, c.f. J.L. Kelley, *General Topology*.

[3] We use the notation $\mathcal{P}(X)$, or sometimes 2^X, to denote the power set of X, i.e. the set of all subsets of X.

Remarks 2.1.

(1) The first three axioms on neighborhood have clear meanings. The fourth one, (N4), gives a relation between neighborhoods of different points and can be regarded as a replacement of triangle inequality for the metric structure.
(2) Such a structure was first introduced by Hausdorff in 1912. [4] His goal was to define a very general notion of space that includes $\mathbb{R}^n$, Riemann surfaces, infinitely dimensional spaces, or spaces consisting of curves or functions. He gave two advantages of introducing such a general notion: to simplify theories, and to prevent us from "illegally" using intuition.

Definition 2.2. A *neighborhood structure* $\mathscr{N}$ on a set X is a map

$$\mathscr{N} : X \to \mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\}$$

satisfying axioms (N1)-(N4).

One can call an abstract set X together with a neighborhood structure $\mathscr{N}$ a *(neighborhood) topological space.* Although it looks very complicated, it turns out that it is equivalent to the usual definition of topological space via open sets that we will give below.

¶ Topology via interior structure.

Given a neighborhood structure $(X, \mathscr{N})$, how do we get the conception of open sets in X? Recall that in mathematical analysis, a set is open if and only if every point in the set is an interior point of the set. We can first define the conception of "interior" in neighborhood topological space:

Definition 2.3. Let $\mathscr{N}$ be a neighborhood structure on X. For any subset $A \subset X$, its *interior* is defined to be

$$\mathrm{Int}(A) := \{x \in A \mid A \in \mathscr{N}(x)\}.$$

Using definition and axioms (N1)-(N4) one can easily check

(I 1) $\mathrm{Int}(A) \subset A$.
(I 2) $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$.
(I 3) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.
(I 4) $\mathrm{Int}(X) = X$.

It turns out that the "interior structure", i.e. a map

$$I : \mathcal{P}(X) \to \mathcal{P}(X)$$

[4] However, the axioms that Hausdorff proposed are a little bit different from the ones above: He requires an additional separation axiom that if $x \neq y$, then there exists $N \in \mathscr{N}(x)$ and $M \in \mathscr{N}(y)$ so that $N \cap M = \emptyset$. Such a separation axiom is called *Hausdorff property* and will be studied later in this course.

satisfying (I 1)-(I 4) is also a structure which is equivalent to the neighborhood structure.

Both "neighborhood axioms" (N1)-(N4) and "interior axioms" (I 1)-(I 4) can be used to define topology on a set. Although it is conceptionally easier to understand the relation between neighborhoods and continuity, neighborhood axioms or interior axioms are not easy to use in practice. The much-easier-to-use way to define topology is via open sets.

¶ Topology via open sets.

Motivated by the conception of open sets in metric spaces (as sets all of whose points are interior points), and given the conception of "interior" above, it is not hard to give the following definition of open sets in $(X, \mathscr{N})$:

Definition 2.4. In a neighborhood topological space $(X, \mathscr{N})$, a set U is said to be *open* if $U \in \mathscr{N}(x)$ for any $x \in U$, or equivalently, if $\mathrm{Int}(U) = U$.

Given $(X, \mathscr{N})$, if we let

$$\mathscr{T} = \{U \subset X \mid U \text{ is open}\}$$

be the set of all open sets in $(X, \mathscr{N})$, one can check:

(O1) $\emptyset \in \mathscr{T}, X \in \mathscr{T}$.
(O2) If $U_1, U_2 \in \mathscr{T}$, so is $U_1 \cap U_2$.
(O3) If $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}$, then $\cup_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}$.

Conversely, given a collection $\mathscr{T}$ satisfying (O1)-(O3), if for any $x \in X$ we let

$$\mathscr{N}(x) = \{N \subset X : \exists U \in \mathscr{T} \text{ s.t. } x \in U \text{ and } U \subset N\},$$

then one can check: $\mathscr{N}(x)$ satisfies (N1), (N2), (N3), (N4). In today's PSets you will be asked to prove the equivalence between (O1)-(O3) and (N1)-(N4) [which is more subtle than proving "(O1)-(O3) $\Rightarrow$ (N1)-(N4)" and "(O1)-(O3) $\Rightarrow$ (N1)-(N4)"].

Since the axioms (O1)-(O3) are much easier to use, we take them as the definition of a topology:[5]

Definition 2.5 (Topology)**.** A *topology* on a set X is a collection $\mathscr{T} \subset \mathcal{P}(X)$ of subsets of X which satisfies (O1), (O2) and (O3). We call such a pair $(X, \mathscr{T})$ a *topological space.*

Since the conception of neighborhoods is so important, we defined it in a topological space formally as follows:

Definition 2.6. Let $(X, \mathscr{T})$ be a topological space. A set $N \subset X$ is called a *neighborhood* of x if there exists an open set $U \in \mathscr{T}$ so that $x \in U \subset N$.

[5]This definition of topology via open sets first appeared in the book "Topology" written by Alexandroff and Hopf in 1935.

¶ Topology via closed sets.

Of course, given the conception of open sets, one can define closed sets:

Definition 2.7. A set F in a topological space $(X, \mathscr{T})$ is *closed* if its complementary $F^c = X \setminus F$ is open.

It is a trivial exercise to convert the "open sets axioms" to "closed sets axioms":

(C1) Both $\emptyset$ and X are closed.
(C2) If U_1, U_2 are closed, so is $U_1 \cup U_2$.
(C3) If U_α are closed for all $\alpha \in \Lambda$, so is $\cap_{\alpha\in\Lambda} U_\alpha$.

¶ Examples of topological spaces.

In what follows we give some examples of topologies.

Example 2.8. (The metric topology) Let (X, d) be any metric space. Let

$$\mathscr{T}_{metric} = \{U \subset X \mid \forall x \in U, \exists r > 0 \text{ s.t. } B(x,r) \subset U\}.$$

Then $\mathscr{T}_{metric}$ is a topology on X. It is called the metric topology.

Example 2.9. (The trivial and discrete topologies) Let X be any set. On X one can always define two "extremal" topologies:

(1) The discrete topology
$$\mathscr{T}_{discrete} = \mathcal{P}(X) = \{Y \mid Y \subset X\}.$$
It is the metric topology associated to the discrete metric on X.

(2) The trivial topology (also called the "indiscrete topology")
$$\mathscr{T}_{trivial} = \{\emptyset, X\}.$$
It is NOT a metric topology for any X with more than one element.

Note for any topology $\mathscr{T}$ on X, we always have

$$\mathscr{T}_{trivial} \subset \mathscr{T} \subset \mathscr{T}_{discrete}$$

Definition 2.10. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on X. we say $\mathscr{T}_1$ is *weaker than*[6] $\mathscr{T}_2$, or equivalently, $\mathscr{T}_2$ is *stronger than* $\mathscr{T}_1$, if $\mathscr{T}_1 \subset \mathscr{T}_2$.

It follows that on any set X, $\mathscr{T}_{trivial}$ is the weakest topology, while $\mathscr{T}_{discrete}$ is the strongest topology.

Remark 2.11. It is possible that two different topologies on X are NOT comparable. For example, the Euclidean topology and the cocountable topology in next example on $\mathbb{R}$ are not comparable.

[6]Some authors use the word *coarser than* instead of "weaker than", and use the word *finer than* instead of "stronger than".

Example 2.12. (The cofinite and cocountable topologies) Let X be any infinite set.

(1) The cofinite topology

$$\mathscr{T}_{cofinite} = \{A \subset X \mid A = \emptyset \text{ or } A^c = X \setminus A \text{ is a finite set}\}.$$

- $\emptyset \in \mathscr{T}; X \in \mathscr{T}$ since $X^c = \emptyset$ is finite.
- If $A, B \in \mathscr{T}$, and $A, B \neq \emptyset$. Then A^c, B^c are finite. So $(A \cap B)^c = A^c \cup B^c$ is finite.
- If $A_\alpha \in \mathscr{T}$ and at least one $A_{\alpha_1} \neq \emptyset$, then $(\cup_\alpha A_\alpha)^c = \cap_\alpha A_\alpha^c \subset A_{\alpha_1}^c$ is finite.

(2) Similarly one can define the cocountable topology (for any set X with uncountably many elements):

$$\mathscr{T}_{cocountable} = \{A \subset X \mid A = \emptyset \text{ or } A^c \text{ is at most countable }\}.$$

[Please check that it is a topology]

Example 2.13. (The Zariski topology) Let $X = \mathbb{C}^n$. Let $R = \mathbb{C}[z_1, \cdots, z_n]$, i.e. the polynomial ring of n-variables with complex coefficients. Define

$$\mathscr{T}_{Zariski} = \{U \subset \mathbb{C}^n \mid \exists f_1, \cdots, f_m \in R \text{ s.t. } U^c = \text{ common zeroes of } f_1, \cdots, f_m\}.$$

One can prove that it is a topology. More generally one can define the Zariski topology on any ring. $\rightsquigarrow$ algebraic geometry.

Example 2.14. (The Sorgenfrey topology) Let $X = \mathbb{R}$. Consider

$$\mathscr{T}_{Sorgenfrey} = \{U \subset \mathbb{R} \mid \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } [x, x + \varepsilon) \subset U\}.$$

Then one can check that it is a topology.

¶ New topological spaces from old.

As in the case of abstract metric spaces, one can construct new topologies from old ones via the standard "restriction to subset" operation and via the "Cartesian product" operations:

(1) (The subspace topology) Let $(X, \mathscr{T})$ be a topological space, and $Y \subset X$ a subset. Then the collection

$$\mathscr{T}_Y := \{U \cap Y \mid U \in \mathscr{T}\}$$

form a topology on Y. It is called the *subspace topology.*

Remark 2.15. If (X, d_X) is a metric space and $Y \subset X$, then "the subspace topology on Y induced from the metric topology on X" coincides with "the metric topology of (Y, d_Y), where d_Y is the induced subspace metric".

(2) (The product topology) Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. Then

$$\mathscr{T}_{X\times Y} := \{W \subset X \times Y \mid \forall (x, y) \in W, \exists U \in \mathscr{T}_X \text{ and } V \in \mathscr{T}_Y \text{ s.t. } (x, y) \in U \times V \subset W\}$$

is a topology on $X \times Y$. It is called the *product topology.*

Remark 2.16. For metrics spaces, “the metric topologies induced from the various product metrics on $X \times Y$ (c.f. Lecture 2)” are all the same, and coincide with “the product topology of the metric topologies on each component”!