Topology (H) Lecture 4
Lecturer: Zuoqin Wang
Time: March 18, 2021

CONVERGENCE AND CONTINUITY

1. Convergence in topological spaces

¶ Convergence.

As we have mentioned, topological structure is created to extend the conceptions of convergence and continuous map to more general setting. It is easy to define the conception of convergence of a sequence in any topological spaces.Intuitively, $x_n \to x_0$ means "for any neighborhood N of x_0, eventually the sequence x_n's will enter and stay in N". Translating this into the language of open sets, we can write

Definition 1.1 (convergence)**.** Let $(X, \mathscr{T})$ be a topological space. Suppose $x_n \in X$ and $x_0 \in X$. We say x_n converges to x_0, denoted by $x_n \to x_0$, if for any neighborhood A of x_0, there exists $N > 0$ such that $x_n \in A$ for all $n > N$.

Remark 1.2. According to the definition of neighborhood, it is easy to see that $x_n \to x_0$ in $(X, \mathscr{T})$ if and only if for any open set U containing x_0, there exists $N > 0$ such that $x_n \in U$ for all $n > N$.

To get a better understanding, let's examine the convergence in simple spaces:

Example 1.3. (Convergence in the metric topology) The convergence in metric topology is the same as the metric convergence: $x_n \to x_0 \Longleftrightarrow \forall \varepsilon > 0$, $\exists N > 0$ s.t. $d(x_n, x_0) < \varepsilon$ for all $n > N$.

Example 1.4. (Convergence in the discrete topology) Since every open ball $B(x, 1) = \{x\}$, it is easy to see $x_n \to x_0$ if and only if there exists N such that $x_n = x_0$ for all $n > N$. In other words, only "eventually constant" sequences converge.

Example 1.5. (Convergence in the trivial topology) Since the only non-empty open set is the set X, any sequence $x_n \in X$ converges and any point $x_0 \in X$ is a limit! In particular, the limit of a convergent sequence is NOT unique![1]

Example 1.6. (Convergence in the cofinite topology) To explore the convergence with respect to the cofinite topology, let's suppose $x_n \to x_0$. Then by definition, for any open neighborhood U of x_0, there exists N such that $x_n \in U$ for $n > N$. This holds if and only if for any $x \neq x_0$, there are at most finitely many $i \in \mathbb{N}$ such that $x_i = x$. So the convergence is very subtle. For example,

[1] Don't worry too much about such bad behaviors. We will see later that under suitable assumptions on the topology, the limit of any convergent sequence is unique.

- If x_n's are all distinct, then $x_1, x_2, \cdots$ converges to any x_0.
- Sequences like $x_0, x_1, x_0, x_2, x_0, ...$ with distinct x_n's will converge, with the unique limit x_0.
- Sequences like $x_1, x_2, x_1, x_2, ...$ do not converge.

Example 1.7. (Convergence in the cocountable topology) Let X be an uncountable set, equipped with the cocountable topology[2]. Suppose $x_n \to x_0$. Take $U = X \setminus \{x_n | x_n \neq x_0\}$. Then U is an open neighborhood of x_0. By definition, there exists $N > 0$ such that $x_n \in U$ for $n > N$, i.e. $x_n = x_0$ for all $n > N$. In other words, only "eventually constant" sequences converge. [So the convergence in the $(X, \mathscr{T}_{cocountable})$ coincides with the convergence in $(X, \mathscr{T}_{discrete})$!]

¶ The pointwise convergence topology.

Let $X = \mathcal{M}([0,1], \mathbb{R})$ be the spaces of all functions (not necessarily continuous) on $[0,1]$. In X we can define pointwise convergence as usual:

$$f_n \to f \text{ if } f_n(x) \to f(x), \forall x \in [0,1].$$

It turns out that we can equip X with a suitable topology, $\mathscr{T}_{p.c.}$, so that the pointwise convergence is exactly the convergence in the topological space $(X, \mathscr{T}_{p.c.})$.

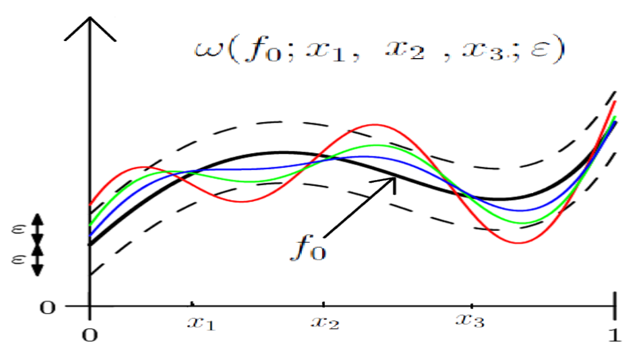
The topology $\mathscr{T}_{p.c.}$ is defined to be

$$\mathscr{T}_{p.c.} = \{U \subset X \mid \forall f_0 \in U, \exists x_1, \cdots, x_n \in [0,1] \text{ and } \varepsilon > 0, \text{ such that } U \supset \omega(f_0; x_1, \cdots, x_n; \varepsilon)\}$$

where

$$\omega(f_0; x_1, \cdots, x_n; \varepsilon) := \{f \in X \bigm| |f(x_i) - f_0(x_i)| < \varepsilon, 1 \le i \le n\}.$$

is the set of functions f's that are ε-close to f_0 at the points $x_1, \cdots, x_n$.



Check. • $\emptyset, X \in \mathscr{T}$

[2]Note: If X is a countable set, we automatically have $\mathscr{T}_{cocountable} = \mathscr{T}_{discrete}$. But for an uncountable set X, $\mathscr{T}_{cocountable}$ is strictly weaker than $\mathscr{T}_{discrete}$.

- If $U_1, U_2 \in \mathscr{T}$, $f_0 \in U_1 \cap U_2$. Then
$$U_1 \supset \omega(f_0; x_1, \cdots, x_n; \varepsilon_1)$$
$$U_2 \supset \omega(f_0; y_1, \cdots, y_m; \varepsilon_2)$$
$\Rightarrow$ $U_1 \cap U_2 \supset \omega(f_0; x_1, \cdots, x_n, y_1, \cdots, y_m; \min(\varepsilon_1, \varepsilon_2))$.
- If $U_\alpha \in \mathscr{T}$, $f_0 \in \cup_\alpha U_\alpha$, Then $\exists \alpha_0$ s.t. $f_0 \in U_{\alpha_0}$.
$\Rightarrow \omega(f_0; x_1, \cdots, x_n; \varepsilon) \subset U_{\alpha_0}$
$\Rightarrow \omega(f_0; x_1, \cdots, x_n; \varepsilon) \subset \cup_\alpha U_\alpha$ i.e. $\cup_\alpha U_\alpha$ is open.

□

¶ Pointwise convergence as a topological convergence.

Fact: The usual pointwise convergence in $X = \mathcal{M}([0,1], \mathbb{R})$ coincides the (topological) convergence in $(X, \mathscr{T}_{p.c.})$.

Proof. Suppose $f_n \to f$ pointwise, and let $U \subset X$ be an open set in $\mathscr{T}_{p.c.}$ with $f \in U$. Then $\exists x_1, \cdots, x_m \in [0,1]$ and $\varepsilon > 0$ s.t.
$$\omega(f; x_1, \cdots, x_m; \varepsilon) \subset U.$$
Since $f_n \to f$ pointwise, we have $f_n(x_i) \to f(x_i), 1 \le i \le m$. So there exists N such that for any $n > N$,
$$f_n \in \omega(f; x_1, \cdots, x_m; \varepsilon) \subset U,$$
i.e. $f_n \to f$ in $(X, \mathscr{T}_{p.c.})$.

Conversely, suppose $f_n \to f$ in $(X, \mathscr{T}_{p.c.})$. For any $x \in [0,1]$, we take $U = \omega(f, x, \varepsilon)$. Then there exists N s.t.
$$f_n \in U, \quad \forall n > N,$$
i.e. $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$. This is the same as saying $f_n(x) \to f(x)$, so $f_n \to f$ pointwise. □

Remark 1.8. We will see later that there exists no metric d on X so that the pointwise convergence is a metric convergence in (X, d). This gives another reason why we need to introduce general topological spaces instead of working only in metric spaces.

2. Continuity of maps between topological spaces

¶ Continuous maps between topological spaces.

As we have explained in Lecture 1, topological structure is the structure that can be used to define continuity for maps. There are two different ways to do so: either use convergence, or use the topological structure itself (namely open sets, closed sets, neighborhoods). Unfortunately the two methods give two different results.

Let's first define continuity via convergent sequences, which match our intuition:

Definition 2.1. We say a map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is

(1) *sequentially continuous at* x_0 if for any convergent sequence $x_n \to x_0$ in X, one has $f(x_n) \to f(x_0)$ in Y,
(2) *sequentially continuous* if it is sequentially continuous everywhere.

You may have noticed that we used the word "sequentially", so that it can be distinguished with continuous maps in general.

To define continuous maps via the topological structure itself (namely using open sets/closed sets/neighborhoods etc), we recall from Lecture 3 a map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is continuous at a point x_0 if and only if the pre-image $f^{-1}(B)$ of any neighborhood B of $f(x_0)$ in the target space Y is a neighborhood of x in the source space,

Inspired by this property, we may define

Definition 2.2. We say a map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is

(1) *continuous at a point* x_0 if the pre-image $f^{-1}(B)$ of any neighborhood B of $f(x_0)$ in Y is a neighborhood of x_0 in X,
(2) a *continuous map* if it is continuous at any point.

From definition we can easily prove

Proposition 2.3. *Let X, Y, Z be topological spaces.*

(1) If $f : X \to Y$ is continuous at x_0 and $g : Y \to Z$ is continuous at $f(x_0)$, then $g \circ f : X \to Z$ is continuous at x_0.
(2) If $f : X \to Y$ is sequentially continuous at x_0 and $g : Y \to Z$ is sequentially continuous at $f(x_0)$, then $g \circ f : X \to Z$ is sequentially continuous at x_0.

Proof. If both f and g are continuous, then for any neighborhood C of $g(f(x_0))$ in Z, $g^{-1}(C)$ is a neighborhood of $f(x_0)$ in Y and thus $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ is a neighborhood of x in X. It follows that $g \circ f$ is continuous at x_0.

If both f and g are sequentially continuous, then for $x_n \to x_0$ in X, we have $f(x_n) \to f(x_0)$ and thus $g(f(x_n)) \to g(f(x_0))$. So $g \circ f$ is sequentially continuous. □

As a consequence, we see that the composition of (sequentially) continuous maps is still (sequentially) continuous.

¶ Sequentially continuity v.s. continuity.

In metric spaces, sequentially continuity and continuity are equivalent. For topological spaces, we have

Proposition 2.4. *If $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous at x_0, then it is also sequentially continuous at x_0. In particular, any continuous map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is sequentially continuous.*

Proof. Suppose $x_n \to x_0$. Take any neighborhood B of $f(x_0)$ in Y. Then by continuity, $f^{-1}(B)$ is a neighborhood of x_0. Since $x_n \to x_0$, there exists $N > 0$ so that $x_n \in f^{-1}(B)$ for $n > N$. It follows that $f(x_n) \in B$ for all $n > N$, i.e. $f(x_n) \to f(x_0)$. So f is sequentially continuous at x_0. □

However, the converse is not true.

Example 2.5. Consider the identity map

$$\mathrm{Id} : (\mathbb{R}, \mathscr{T}_{cocountable}) \to (\mathbb{R}, \mathscr{T}_{discrete}), \quad x \mapsto x.$$

Then Id is sequentially continuous, since as have have seen, a sequence converges in $(\mathbb{R}, \mathscr{T}_{cocountable})$ if and only if it converges in $(\mathbb{R}, \mathscr{T}_{discrete})$ (and to the same limit). However, Id is not continuous at any point: for any $x \in \mathbb{R}$, the interval $[x-1, x+1]$ is an open neighborhood of x in $(\mathbb{R}, \mathscr{T}_{discrete})$, but not a neighborhood of x in $(\mathbb{R}, \mathscr{T}_{cocountable})$.

¶ Global continuity via open sets.

We showed in Theorem 1.6 in Lecture 3 that $f : (X, d_X) \to (Y, d_Y)$ is a continuous map if and only if the pre-image $f^{-1}(V)$ of any open set V in Y is an open set in X. By repeating the proof word by word, one can easily write down a proof of the following characterization of global continuity via open sets:

Proposition 2.6. *Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a map. Then f is continuous if and only if for any $V \in \mathscr{T}_Y$, one has $f^{-1}(V) \in \mathscr{T}_X$.*

By taking complementary, we have

Proposition 2.7. *Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous if and only if for any closed set F in Y, the pre-image $f^{-1}(F)$ is closed in X.*

Proof. Note that $f^{-1}(F)$ is closed if and only if $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$ is open. So the conclusion follows. □

Remark 2.8. In many cases a fact described by open sets has a "dual description" via closed sets. We call this principle the *open-closed duality.*

¶ Open/closed maps.

So under continuous maps, the pre-image of an open set is open, and the pre-image of any closed maps is closed. But in general, (write an example for each!)

- the image of an open set under a continuous map needn't be open,
- the image of a closed set under a continuous map needn't be closed.

Definition 2.9. Let X, Y be topological spaces. A map $f : X \to Y$ is called

- an *open map* if for any open set U in X, $f(U)$ is open in Y.
- a *closed map* if for any closed set F in X, $f(F)$ is closed in Y.

Although it seems that open/closed maps are "more natural", they are not as important/convenient as continuous maps in topology. Here is one reason:
We always have

$$f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha), \quad f^{-1}(\bigcap_\alpha B_\alpha) = \bigcap_\alpha f^{-1}(B_\alpha), \quad f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$

But in general, we only have

$$f(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f(A_\alpha), \quad f(\bigcap_\alpha A_\alpha) \subset \bigcap_\alpha f(A_\alpha), \quad f(X \setminus A) \supset f(X) \setminus f(A).$$

However, open/closed maps do appear in some other branches of mathematics and plays a very important role. For example,

- One of the most important theorems in functional analysis, the open mapping theorem, claims that every surjective continuous linear operator between Banach spaces is an open map.
- In complex analysis, there is also an open mapping theorem which states that any non-constant holomorphic function defined on a connected open subset of the complex plane is an open map.
- We will prove later in this course the following *Brouwer's invariance of domain theorem*: if $U \subset \mathbb{R}^n$ is an open set, then any injective continuous map $f : U \to \mathbb{R}^n$ is an open map.

¶ Examples of continuous maps.

We give some examples of continuous maps.

Example 2.10. Any constant function $f : X \to Y$ is continuous.

> Reason: Suppose $f(x) \equiv y_0 \in Y$, and let U be any open set in Y. Then
> Case 1: $y_0 \in U \Rightarrow f^{-1}(U) = X$ is open in X.
> Case 2: $y_0 \notin U \Rightarrow f^{-1}(U) = \emptyset$ is open in X.
> So f is continuous.

Remark 2.11. This explains why we always assume $\emptyset, X$ to be open in any topology: if not, constant functions could be discontinuous!

Example 2.12. From Proposition 2.6 and definitions we immediately get

- Any map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_{trivial})$ is continuous.
- Any map $f : (X, \mathscr{T}_{discrete}) \to (Y, \mathscr{T}_Y)$ is continuous.
- The identity map $\mathrm{Id} : (X, \mathscr{T}_2) \to (X, \mathscr{T}_1)$ is continuous if and only if $\mathscr{T}_1 \subset \mathscr{T}_2$, i.e. $\mathscr{T}_1$ is weaker than $\mathscr{T}_2$.
- If $A \subset X$, then the inclusion map $\iota : A \hookrightarrow X$ is continuous.
- If $f : X \to Y$ is continuous, $A \subset X$, then $f|_A : A \to Y$ is continuous.

Example 2.13. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *right continuous* at x_0 if

$$\lim_{x \to x_0+} f(x) = f(x_0).$$

Fact. $f : \mathbb{R} \to \mathbb{R}$ is right continuous at $x_0 \iff f : (\mathbb{R}, \mathscr{T}_{Sorgenfrey}) \to (\mathbb{R}, \mathscr{T}_{usual})$ is continuous at x_0. [3]

Proof. Left as exercise. □

Example 2.14. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. Let $(X \times Y, \mathscr{T}_{X\times Y})$ be the product topological space. Then the projections

$$\pi_X : X \times Y \to X, \quad (x, y) \mapsto x$$

$$\pi_Y : X \times Y \to X, \quad (x, y) \mapsto y$$

are continuous maps and are open maps.

Proof. We only prove the conclusions for π_X, since the proof for π_Y is similar. π_X is continuous since

$$\forall U \in \mathscr{T}_X, \pi_X^{-1}(U) = U \times Y \in \mathscr{T}_{X\times Y}.$$

It is an open map because for any open set $W \in X \times Y$ and any $x \in \pi_X(W)$, there exists a point $(x, y) \in W$. By definition of the product topology, there exists open sets $U \ni x$ in X and $V \ni y$ in Y such that $(x, y) \in U \times V \subset W$. It follows that $x \in U \subset \pi_X(W)$. So $\pi_X(W)$ is open in X, i.e. π_X is an open map. □

Remark 2.15. The projection map need not be closed. For example, the projection of the closed set $\{(x, 1/x) \mid x > 0\} \subset \mathbb{R}^2$ onto $\mathbb{R}$ is $(0, +\infty)$ which is not closed in $\mathbb{R}$.

¶ Homeomorphism.

Using continuous maps, we can define equivalence of topological spaces.

Definition 2.16. We say topological spaces X and Y are *homeomorphic*, denoted by $X \simeq Y$, if there exists an invertible map $f : X \to Y$ such that both f and f^{-1} are continuous. The map f is called a *homeomorphism* between X and Y.

Example 2.17. With the usual topology,

(1) $(0, 1) \simeq \mathbb{R}$. [Can you explicitly construct an homeomorphism?]
(2) $S^n - \{\text{the north pole}\} \simeq \mathbb{R}^n$.
(3) $[0, 1] \not\simeq (0, 1) \not\simeq [0, 1) \not\simeq S^1 \not\simeq \mathbb{R}^2$.

Proposition 2.18. *Homeomorphism is an equivalent relation among topological spaces.*

Proof. We have

- $X \simeq X$: since $\mathrm{Id} : (X, \mathscr{T}_X) \to (X, \mathscr{T}_X)$ is a homeomorphism.
- $X \simeq Y \implies Y \simeq X$: If $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$.

[3] So the Sorgenfrey topology is also called “the right continuous topology”.

- $X \simeq Y, Y \simeq Z \Longrightarrow X \simeq Z$: If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is bijective. Moreover, by Proposition 2.3, both $g \circ f$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ are continuous.

So the conclusion follows. □

Remark 2.19. We will regard homeomorphic topological spaces as the same space. We will say a property is a *topological property* if it is preserved under a homeomorphism.

Other than being continuous and bijective, it is clear from definition that homeomorphisms must be both an open map and a closed map. Conversely,

Proposition 2.20. *Let $f : X \to Y$ be a bijective continuous map. If f is either open or closed, then f is a homeomorphism.*

Proof. This follows from a simple observation: if f is invertible, then f^{-1} is continuous if and only if f is open (and if and only if f is closed). □

As in the metric case, we can define the conception of a topological embedding:

Definition 2.21. Let $f : X \to Y$ be an injective continuous map. We say f is a *topological embedding* if f is a homeomorphism from X to $f(X) \subset Y$ (endowed with the subspace topology).

¶ Compatibility: Topological groups and topological vector spaces.

We can also use continuity to define compatibility of different structures with topological structure. For example,

Definition 2.22. A topological group is a group G with a topological structure[4] so that the groups operations

$$m : G \times G \to G, \quad (g_1, g_2) \mapsto m(g_1, g_2) := g_1 \cdot g_2$$

and

$$i : G \to G, \quad g \mapsto i(g) := g^{-1}$$

are continuous maps. [Here $G \times G$ is endowed with the product topology.]

Similarly one can define *topological rings*, *topological fields* etc.

Topological groups (and their smooth analogues, *Lie groups*) are used widely in mathematics to describe continuous symmetries. Here are some examples:

Example 2.23.

(1) (Not interesting)Any group G, with the discrete topology, is a topological group.
(2) $\mathbb{R}$ and $\mathbb{C}$, with the usual group structure and the usual topology, are topological groups (and in fact are topological fields).

[4] In the definition of topological groups, some authors will require the topology on G to satisfy further separation properties like T_1 or T_2.

(3) S^1, $\mathbb{R}^n$, $\mathbb{T}^n := (S^1)^n$ are topological groups (with the usual structures).
(4) Matrix groups $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ etc are topological groups (with the usual structures).
(5) Examples in (2), (3), (4) are in fact Lie groups. Here is a topological group which is not a Lie group: $\mathbb{Q}$, with the usual structures, is a topological group.

In functional analysis where people study analysis on (usually infinitely dimensional) vector spaces, it is crucial that the topological structure of the vector space is compatible with the vector space structure:

Definition 2.24. A *topological vector space* is a vector space X over $\mathbb{R}$ or $\mathbb{C}$ (or a topological field $\mathbb{K}$) that is endowed with a topology[5] such that the vector addition map

$$+ : X \times X \to X, \quad (x, y) \mapsto x + y$$

and the scalar multiplication map

$$\bullet : \mathbb{K} \times X \to X, \quad (\lambda, x) \mapsto \lambda x$$

are continuous maps. [Here both $X \times X$ and $\mathbb{K} \times X$ are endowed with the product topology.]

Note that a topological vector space is automatically a topological group.

Example 2.25.

(1) $\mathbb{R}^n$, $\mathbb{C}^n$, with the usual structures.
(2) Warning: $\mathbb{R}^n$ is NOT a topological vector space when endowed with the discrete topology. [Although the vector addition is still continuous, the scalar multiplication is no longer continuous.]
(3) {Hilbert spaces} $\subset$ {Banach spaces} $\subset$ {normed vector spaces} $\subset$ {Fréchet spaces} $\subset$ {locally convex topological vector spaces} $\subset$ {topological vector spaces}

[5] Again, in the definition of topological vector spaces, some authors will require the topology on G to satisfy further separation properties like T_1.