Topology (H) Lecture 6
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Time: March 25, 2021

THE QUOTIENT TOPOLOGY

1. The quotient topology

¶ The quotient topology.

Last time we introduced several abstract methods to construct topologies on abstract spaces (which is widely used in point-set topology and analysis). Today we will introduce another way to construct topological spaces: the quotient topology.

In fact the quotient topology is not a brand new method to construct topology. It is merely a simple special case of the co-induced topology that we introduced last time. However, since it is very concrete and "visible", it is widely used in geometry and algebraic topology. Here is the definition:

Definition 1.1 (The quotient topology)**.**

(1) Let $(X, \mathscr{T}_X)$ be a topological space, Y be a set, and $p : X \to Y$ be a surjective map. The co-induced topology on Y induced by the map p is called the *quotient topology* on Y. In other words,

$$\text{a set } V \subset Y \text{ is open if and only if } p^{-1}(V) \text{ is open in } (X, \mathscr{T}_X).$$

(2) A continuous surjective map $p : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is called a *quotient map*, and Y is called the *quotient space* of X if $\mathscr{T}_Y$ coincides with the quotient topology on Y induced by p.

(3) Given a quotient map p, we call $p^{-1}(y)$ the *fiber* of p over the point $y \in Y$.

Note: by definition, the composition of two quotient maps is again a quotient map.

Here is a typical way to construct quotient maps/quotient topology: Start with a topological space $(X, \mathscr{T}_X)$, and define an equivalent relation $\sim$ on X. Recall that this means

- $x \sim x$;
- $x \sim y \Longrightarrow y \sim x$;
- $x \sim y, y \sim z \Longrightarrow x \sim z$.

Then one gets an abstract space consisting of all equivalence classes

$$Y = X/\sim$$

and a natural projection map

$$p : X \to X/\sim,\ x \mapsto [x].$$

So in this case, each fiber is an equivalence class. Note that the "quotient by a map" description and the "quotient by an equivalence relation" description are equivalent: Given any equivalence relation description, we have a natural projection map as shown above; conversely given any quotient map $f : X \to Y$, we can define an equivalence relation by $x \sim y \iff f(x) = f(y)$ and thus get an equivalence relation description of the same quotient space.

Example 1.2 (The circle). One can regard the circle S^1 as a quotient space via

(1) $S^1 = [0,1]/\{0,1\}$: in other words, the only equivalence is $0 \sim 1$.
(2) $S^1 = \mathbb{R}/\mathbb{Z}$: in other words, we used the equivalence relation

$$x \sim y \iff x - y \in \mathbb{Z}.$$

Example 1.3. Define an equivalence relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \iff x - y \in \mathbb{Q}.$$

Then what is the quotient topology on $X = \mathbb{R}/\sim$? Let $U \subset X$ be an open set. Then $p^{-1}(U)$ is open in $\mathbb{R}$. In particular, there is an interval $(a,b) \subset U$. Since any real number $x \in \mathbb{R}$ is equivalent to some number in (a,b), we must have $p^{-1}(U) = \mathbb{R}$. So the quotient topology on X is the trivial topology.

¶ Universality.

According to the universality of the co-induced topology, namely Proposition 2.8 in Lecture 5 (whose proof is in your PSet), we have

Theorem 1.4 (Universality of quotient topology)**.** *Let X, Y, Z be topological spaces, $p : X \to Y$ be a quotient map, and $f : Y \to Z$ be a map. Then f is continuous if and only if $g = f \circ p$ is continuous. Moreover, the quotient topology on Y is the only topology satisfying this property.*

As a consequence, we have

Corollary 1.5. *If $p : X \to Y$ is a quotient map, $f : X \to Z$ is a continuous map such that f is constant on each fiber. Then the naturally induced map*

$$\bar{f} : Y \to Z, \qquad \bar{f}(y) := f(p^{-1}(y))$$

is continuous.

¶ The real projective space.

Let's give an important example of quotient space: the real projective space. We can give two descriptions.

Example 1.6 (The real projective space).

- On $X = \mathbb{R}^{n+1} - \{0\}$ we can define an equivalence relation
$$x \sim y \iff \exists 0 \neq \lambda \in \mathbb{R} \text{ s.t. } x = \lambda y.$$
The quotient space
$$\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}/\sim$$
(endowed with the quotient topology[1]) is called the real projective space. This gives a geometric explanation of the real projective space:
$$\boxed{\mathbb{RP}^n = \text{the space of all lines in } \mathbb{R}^{n+1} \text{ passing the origin } 0.}$$
- We can also start with the unit sphere $S^n \subset \mathbb{R}^{n+1}$ and define an equivalence relation
$$x \sim y \Leftrightarrow x = \pm y.$$
Since every line in $\mathbb{R}^{n+1}$ passing the origin 0 intersect S^n exactly at two antipodal points, the resulting quotient space are the same.

Note that when $n = 1$, $\mathbb{RP}^1$ is in fact homeomorphic to S^1, since according to the second description, we may start with a half circle and identify the two end points. However, the geometric picture, even in the case $n = 2$ where we may start the construction with a hemisphere, is very complicated:

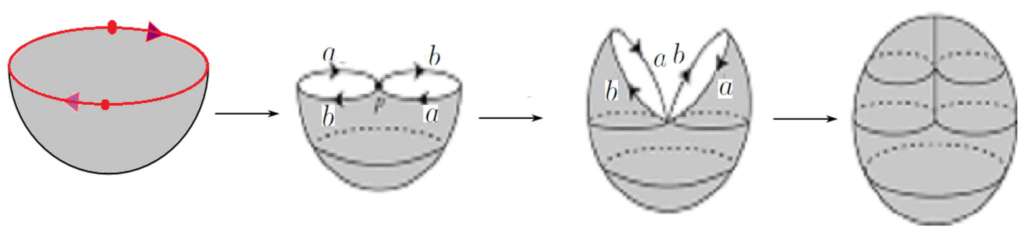

FIGURE 1. A "picture" of $\mathbb{RP}^2$

As one can see, there is a "self-intersection" in the picture. However, the intersection should not exist in a real "picture" of $\mathbb{RP}^2$. In fact, there is no way to embed $\mathbb{RP}^2$ into $\mathbb{R}^3$. It can only be embedded into $\mathbb{R}^4$. Moreover, just like the Möbius band, $\mathbb{RP}^n$ is not orientable for any even number n.

Remark 1.7. Similarly, one can define a topology on the space of complex lines in $\mathbb{C}^{n+1}$ (the complex projective space $\mathbb{CP}^n$). More generally, one can define a topology on the space of k-dimensional vector subspaces of a vector space V (the Grassmannian manifold $Gr(k, V)$[2]). Note that $\mathbb{RP}^n$ is just a special Grassmannian : $\mathbb{RP}^n = Gr(1, \mathbb{R}^{n+1})$.

[1]Here, we endow with $\mathbb{R}^{n+1} - \{0\}$ the standard Euclidean topology.

[2]However, for $k > 1$, $Gr(k, V)$ can not be realized as a quotient space of V. Instead it can be realized as a quotient space of a much larger space, e.g. $\mathrm{GL}(V)$.

¶ Construction: Gluing one point to another in the same space.

In what follows we will introduce many very concrete geometric ways to get quotient spaces from known spaces. The first is :

> Gluing: Let X be a topological space, by *gluing* the points a and b in X, we means: considering the quotient space obtained from the equivalence relation which contains only one non-trivial equivalence: $a \sim b$. Similarly, we may glue a subset A to a subset B in X by identifying each point in A to a specific point in B.

This is widely used in constructing surfaces topologically from planar polygons: just glue boundary line segments using prescribed way.[3]

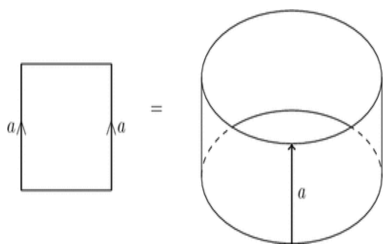

FIGURE 2. The cylinder

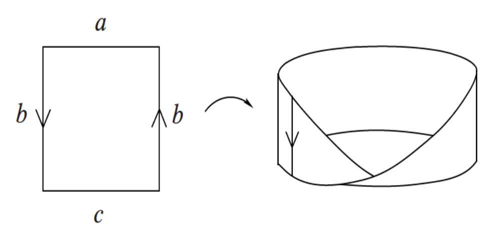

FIGURE 3. The Möbius band

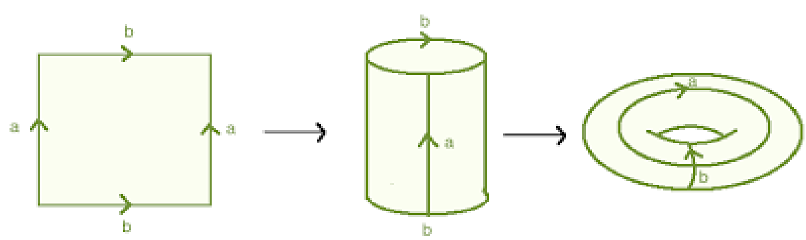

FIGURE 4. The torus

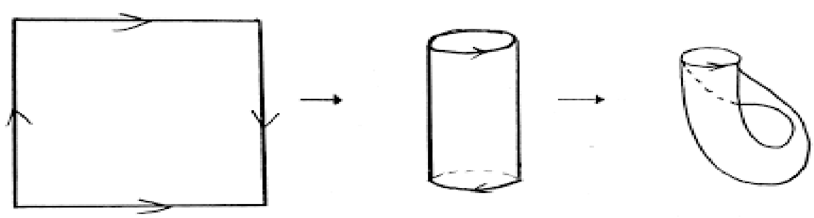

FIGURE 5. The Klein bottle

[3]Note that the Klein bottle can not be embedded into $\mathbb{R}^3$. It is a non-orientable surface.

It turns out that any (compact) surface can be constructed by starting at a suitably-chosen polygon and attaching its boundary edges in suitable way. Here is a more complicated one:

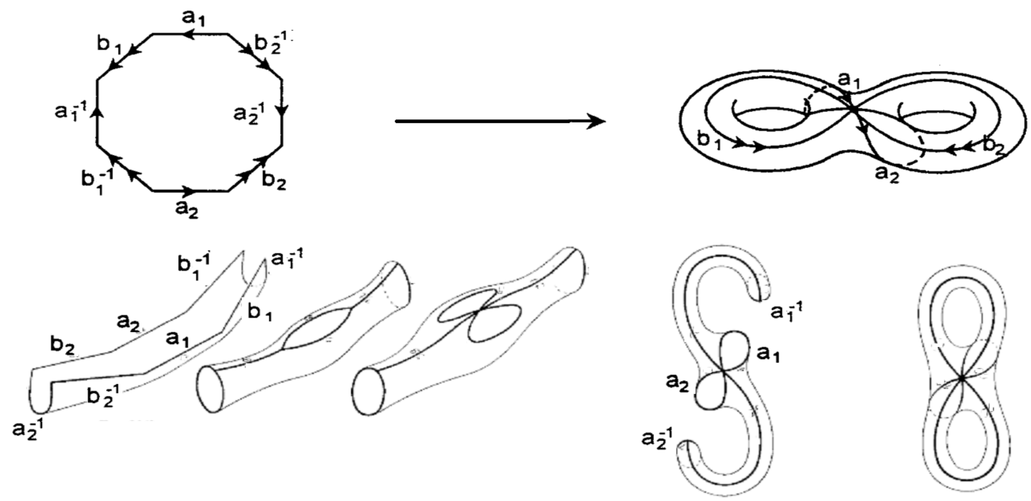

FIGURE 6. The 2-torus

At the end of this semester, we will use such polygonal presentation to prove the classification theorem of compact surfaces.

¶ Construction: Attaching space (adjunction space).

We may attach one space to another along a given map:

> Attaching space: Let X, Y be topological spaces, and $A \subset Y$ a subspace, and $f : A \to X$ a continuous map. Then the attaching space $X \cup_f Y$ is formed by taking the disjoint union of X and Y and identifying each $a \in A$ with $f(a) \in X$, i.e.
>
> $$X \cup_f Y = X \sqcup Y/\{a \sim f(a)\}.$$

There are two special cases that we are going to use later:

(1) (The wedge sum) Given two topological spaces X and Y, the *wedge sum* $X \vee Y$ of X and Y is formed by attaching one point in X to one point in Y:

$$X \vee Y = X \sqcup Y/\{x_0 \sim y_0\}.$$

More generally, given a family of spaces X_α, with a point $x_\alpha \in X_\alpha$ chosen, we may form the *wedge sum* $\bigvee_\alpha X_\alpha$ by attaching all X_α's at the point x_α's.

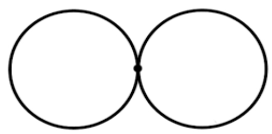

FIGURE 7. $S^1 \vee S^1$

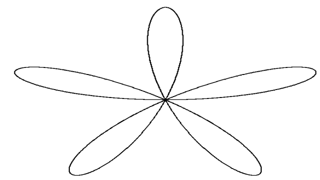

FIGURE 8. $S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$

Remark 1.8. When talking about wedge sum, we are really working on "pointed space" (X, x_0), namely a space with a point x_0 chosen. The wedge sum of (X, y_0) and (Y, y_0) is again a pointed space $(X \vee Y, \{x_0\})$. By this way, when studying the wedge sum of many spaces, we are always attaching the marked points into one point.

(2) (The connected sum) Given two geometric objects A and B that are locally Euclidian ("manifolds"), the *connected sum* $A\#B$ is constructed as follows: one can remove a small ball (disk) from each, and then glue the boundary spheres (circles) so that they are "connected together". [4]

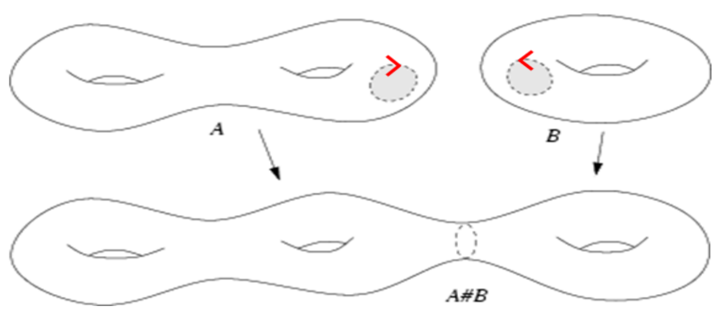

FIGURE 9. The connected sum $A\#B$

In other words,

$$A\#B = (A - D_1) \cup_f (B - D_2),$$

where D_1, D_2 are small disks on A and B respectively, and f is the attaching map that identify ∂D_1 with ∂D_2 as shown in the picture.

¶ Construction: Squeeze a subset into one point.

Next let's consider

[4]This is widely used in constructing new surfaces from given surfaces, or more in constructing new manifolds from given ones in manifold theory.

Squeeze: Let X be a topological space, and $Y \subset X$. We may define an equivalence relation on X by requiring and only requiring $y_1 \sim y_2$ for any $y_1, y_2 \in Y$. In other words, in the quotient space, we "squeeze" all points in Y to one point. For simplicity we just denote the quotient space by X/Y.

For example, we consider the unit disk D in $\mathbb{R}^2$. We can squeeze its boundary circle into one point. What do we get? A sphere S^2! Similarly we may squeeze the boundary sphere S^{n-1} of the unit ball $B(0,1)$ in $\mathbb{R}^n$ to get $\mathbb{S}^n$.

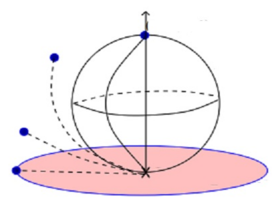

FIGURE 10. Squeeze the boundary circle to get a sphere

As another example: By regarding X as $X \times \{y_0\}$ and regarding Y as $\{x_0\} \times Y$, we can view the wedge sum $X \vee Y$ as a subspace of $X \times Y$. Then one may define the *smash product* of X and Y, denoted by $X \wedge Y$, as

$$X \wedge Y = X \times Y / X \vee Y.$$

¶ Construction: the cone space and the suspension.

Given any topological space X, one may construct the cone space and the suspension of X (used in algebraic topology), both as a quotient space of the cylinder $X \times [0,1]$:

(1) The *cone space* of X, denoted by $C(X)$, is formed by squeezing $X \times \{0\}$ in $X \times [0,1]$ into one point, namely, $C(X) = X \times [0,1]/X \times \{0\}$:

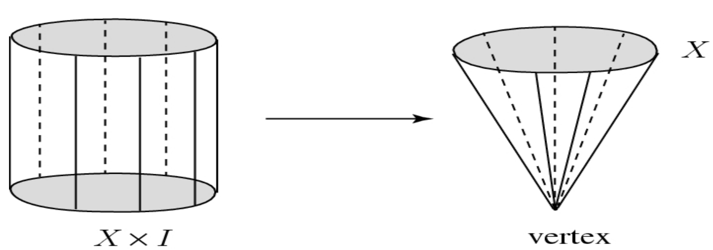

FIGURE 11. The cone space $C(X)$

(2) The *suspension* of X, denoted by $S(X)$, is formed by squeezing $X \times \{0\}$ to one point, and also squeezing all points in $X \times \{1\}$ to another point.

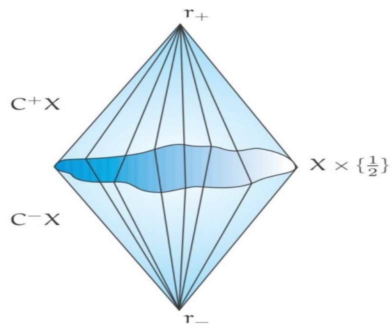

FIGURE 12. The suspension $S(X)$

(3) More generally, given topological spaces X and Y, the *join* of X and Y, sometimes denoted by $X \star Y$, is defined as $X \star Y = X \times Y \times I / \sim$, where $\sim$ is given by

$$(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1), \qquad \forall x, x_1, x_2 \in X; y, y_1, y_2 \in Y.$$

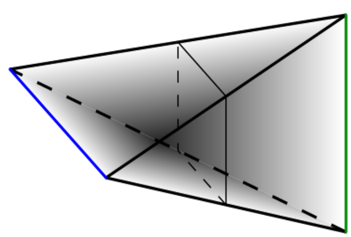

FIGURE 13. The join $X \star Y$

¶ Construction: Mapping cylinder, mapping cone and mapping torus.

One may also study spaces associated to maps:

(1) Given a continuous map $f : X \to Y$, the *mapping cylinder* of f, denoted by

$$M_f = (X \times [0, 1]) \sqcup_{\tilde{f}} Y,$$

is by definition the attaching space of $X \times [0, 1]$ and Y via the map $\tilde{f} : X \times \{0\} \to Y$, $f(x, 0) := f(x)$.

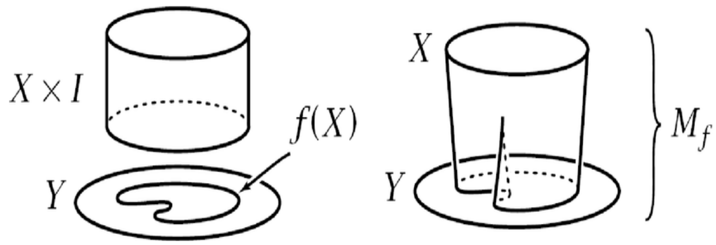

FIGURE 14. Mapping cylinder

(2) Given a continuous map $f : X \to Y$, the *mapping cone* of f, denoted by C_f, is by definition to be the quotient space

$$C_f = (X \times [0, 1]) \sqcup_{\tilde{f}} Y/ \sim,$$

namely the quotient space of the mapping cylinder M_f with respect to the equivalence relation

$$(x_1, 1) \sim (x_2, 1), (x, 0) \sim f(x), \qquad \forall x, x_1, x_2 \in X.$$

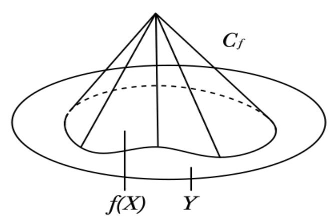

FIGURE 15. Mapping cone

(3) Given a homeomorphism $f : X \to X$, the *mapping torus* of f is defined to be

$$T_f := X \times [0, 1]/(1, x) \sim (0, f(x)).$$

Mapping tori of surface homeomorphisms play a key role in the theory of 3-manifolds and have been intensely studied.

2. Quotient by a group action

¶ The homeomorphism group.

Symmetries play an essential role in all branches of mathematics. The mathematical language of symmetry is group.

Proposition 2.1. *Let X be a topological space and let*

$$\text{Hom}(X) = \{f : X \to X \mid f \textit{ is a homeomorphism}\}.$$

Then under the usual composition of maps, Hom(X) *is a group.*

Proof. This is almost trivial:

- given two homeomorphisms f and g of X, the composition $g \circ f$ is again a homeomorphism of X, and the associativity holds by definition,
- the identity map Id is the identity element in this group,
- the inverse map f^{-1} is a homeomorphism and is the inverse of f in this group.

□

So given any topological space X, we have a god-given group that describes symmetries of X in the category of topology:

Definition 2.2. Hom(X) is called the *homeomorphism group* of X.

Note that for any element $f \in \text{Hom}(X)$, we may say f "acts" on the space X by sending an element $x \in X$ to its image $f(x) \in X$.

¶ The group action.

We define

Definition 2.3. Let G be any group, and X be a space.

(1) An *(left) action*[5] of group G on the space X is a map
$$\alpha : G \times X \to X, \qquad (g, x) \mapsto g \cdot x,$$
so that for any $x \in X$ and any $g, h \in G$,
  - $e \cdot x = x$ for any $x \in X$,
  - $g \cdot (h \cdot x) = (gh) \cdot x$, for any $g, h \in G$ and $x \in X$.

(2) In the case X is a topological space, an *(left) action* of the group G on the topological space X is an action so that for any $g \in G$, the map
$$\tau_g : X \to X, \quad x \mapsto \tau_g(x) := g \cdot x$$
is a continuous map (and thus is a homeomorphism since $(\tau_g)^{-1} = \tau_{g^{-1}}$ is also continuous).

(3) In the case X is a topological space and G is a topological group, we say the action is a *continuous action* if the map α is a continuous map.

Remark 2.4. So an action of G on a topological space X is a group homomorphism
$$\tau : G \to \text{Hom}(X) = \{f : X \to X \mid f \text{ is a homeomorphism}\},$$
i.e. associate to any $g \in G$ a homeomorphism $\tau_g : X \to X$, such that
$$\tau_g \circ \tau_h = \tau_{gh}, \quad \forall g, h \in G.$$

[5]There is also a conception of *right action*, in which we use $x \cdot g$ instead, and replace the second condition by $(x \cdot g) \cdot h = x \cdot (gh)$. The theory for right action is almost the same as left actions.

Note that we may (and will always) assume τ is injective. Otherwise we can always replace G by $G/\ker(\tau)$, which acts on X in the obvious way. Such an action is called a *faithful action*.

¶ Orbits and the orbit space.

Definition 2.5. Given a group action of G on X, the *orbit* of $x \in X$ is the set

$$G \cdot x := \{g \cdot x \mid g \in G.\}$$

We will see many examples below where the orbit is very simple. Here we give an example where the orbit is very complicated:

Example 2.6. Consider S^1 acts on $S^1 \times S^1$ by

$$e^{i\alpha} \cdot (e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\theta+\alpha)}, e^{i(\theta+\sqrt{2}\alpha)}).$$

Then the orbit is a "dense curve" on the torus $S^1 \times S^1$.

We may define an equivalence relation $\sim$ on X by

$$x_1 \sim x_2 \Longleftrightarrow \exists\, g \in G \ \text{ s.t. } x_1 = g \cdot x_2.$$

In other words, two elements in X are equivalent if and only if they lie in the same orbit. It is easy to check that this is an equivalence relation.

Definition 2.7. Given a group action of G on a topological space X, the *orbit space* is defined to be the quotient space $X/G = X/\sim$.

So by definition, the orbit space is "the space of orbits", endowed with the quotient topology.

Example 2.8. Consider the $\mathbb{R}_{>0}$ (as a multiplicative group) action on $\mathbb{R}$ by multiplication, i.e.

$$a \cdot x := ax.$$

Then there are three orbits: $\mathbb{R}_{>0}$, $\{0\}$, $\mathbb{R}_{<0}$. As a result, the orbit space consists of three elements, $\{+, 0, -\}$, and the topology on the orbit space is

$$\{\emptyset, \{+\}, \{-\}, \{+,-\}, \{+,0,-\}\}.$$

¶ Examples.

We list several simple examples of orbit spaces which we will use when we study covering spaces later this semester.

Example 2.9 (S^1 again). $G = \mathbb{Z}$ acts on $X = \mathbb{R}$ via

$$\tau(n)(x) = n + x. \qquad \text{(translation)}$$

$\rightsquigarrow \ \mathbb{R}/\mathbb{Z} \simeq S^1$.

Example 2.10 (S^1 again and again). $G = \mathbb{Z}_n$ acts on $X = S^1 \subset \mathbb{C}$ via

$$\tau(k)(z) = e^{2\pi i k/n} z. \qquad \text{(rotation)}$$

$\rightsquigarrow$ $S^1/\mathbb{Z}_n \simeq S^1$.

Example 2.11 ($\mathbb{RP}^n$ again). $G = \mathbb{Z}_2$ acts on $\widetilde{X} = S^n$ via

$$\tau(1)(x) = x \quad \text{and} \quad \tau(-1)(x) = -x. \qquad \text{(antipodal)}$$

$\rightsquigarrow$ $S^n/\mathbb{Z}_2 \simeq \mathbb{RP}^n$.

Example 2.12 (n-torus). $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ via

$$\tau(m_1, \cdots, m_n)(x_1, \cdots, x_n) = (x_1 + m_1, \cdots, x_n + m_n).$$

$\rightsquigarrow$ $\mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{T}^n \simeq S^1 \times \cdots \times S^1$.

Example 2.13 (Lens space $L(p,q)$). Let p, q be co-prime numbers. We define an action of $G = \mathbb{Z}_p = \{1, e^{2\pi i/p}, \cdots, e^{2\pi i(p-1)/p}\}$ on $X = S^3 \subset \mathbb{C}^2$ via

$$\tau(e^{2\pi ik/p})(z_1, z_2) = (e^{2\pi ik/p} z_1, e^{2\pi ikq/p} z_2).$$

$\rightsquigarrow$ $L(p;q) := S^3/\mathbb{Z}_p$ is known as the *lens space*.

¶ Example: Hopf fibration.

Finally we give an example of a continuous group action (which is not properly discontinuous).

Let's regard the circle group S^1 as

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

and regard the three dimensional sphere S^3 as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

Then we can define an action of S^1 on S^3 via

$$z \cdot (z_1, z_2) := (zz_1, zz_2).$$

Then one has

(1) Each orbit $S^1 \cdot (z_1, z_2)$ is homeomorphic to a circle.
(2) The orbit space S^3/S^1 is homeomorphic to S^2:
A sketch of proof: We let

$$X = \{(z_1, z_2) \in S^3 \mid |z_1| \leq |z_2|\}$$

and

$$Y = \{(z_1, z_2) \in S^3 \mid |z_1| \geq |z_2|\}.$$

Note that both X and Y (and thus $X \cap Y$) are invariant under the S^1-action. So S^3/S^1 can be constructed as gluing X/S^1 and Y/S^1 along the "boundary" $X \cap Y/S^1$ which is a quotient of the torus

$$X \cap Y = (z_1, z_2) \in S^3 \mid |z_1| = |z_2|\}$$

by S^1, and thus is a circle. Now consider X/S^1. We can define a map

$$f : D^2 \to X, z \mapsto \frac{1}{\sqrt{2}}(z, 1)$$

and show that f is a homeomorphism which maps the boundary circle of D^2 to the boundary circle $X \cap Y/S^1$. Similarly Y/S^1 is homeomorphic to a disk whose boundary gets mapped to $X \cap Y/S^1$. As a result, the quotient S^3/S^1 is homeomorphic to the space obtained by gluing two unit disks along their boundary, which is the sphere S^2!

The quotient map $p : S^3 \to S^3/S^1 \simeq S^2$ is known as the *Hopf fibration* and plays an important role in geometry and topology.