Topology (H) Lecture 8
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COMPACTNESS: DEFINITIONS AND BASIC PROPERTIES

1. Compactness: various definitions and examples

¶ Properties of $[0,1]$.

As we have mentioned in Lecture 1, compactness is a generalization of finiteness. The simplest compact sets are finite sets. The next simplest compact set are the bounded closed intervals. Let's compare finite sets, $[0,1]$ and $(0,1)$:

	$X=$ a finite set	$X=[0,1]$	$X=(0,1)$
A continuous function $f: X \to \mathbb{R}$	bounded, and attains its maximal/minimal values.	(*Extremal value property*) bounded, and attains its maximal/minimal values.	could be unbounded, or bounded without extremal value.
A sequence $x_1, x_2, \cdots$ in X	has a constant subsequence $x_{n_1} = x_{n_2} = \cdots = c \in X$.	(*Bolzano-Weierstrass*) has a convergent subsequence $x_{n_1}, x_{n_2}, \cdots \to c \in X$.	$1, \frac{1}{2}, \frac{1}{3}, \cdots$ has no convergent subsequence
An infinite subset $A \subset X$	———	A has a limit point $c \in X$.	$\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ has no limit point
If $X = \bigcup_\alpha U_\alpha$, with U_α open	$\exists U_{\alpha_1}, \cdots, U_{\alpha_k}$ s.t. $X = \bigcup_{i=1}^k U_{\alpha_i}$.	(*Heine-Borel*) $\exists U_{\alpha_1}, \cdots, U_{\alpha_k}$ s.t. $X = \bigcup_{i=1}^k V_{\alpha_i}$.	$\bigcup_{n=1}^\infty (\frac{1}{n}, 1)$ no finite subcovering.
$F_1 \supset F_2 \supset \cdots$ a nested sequence of closed sets	$\bigcap_k X_k \neq \emptyset$	(*Cantor*) $\bigcap_k F_k \neq \emptyset$.	$\bigcap_{k=1}^\infty (0, \frac{1}{k}] = \emptyset$.

Remark 1.1. Why the finiteness/compactness is important? Because you can get global information from local information. (*local-to-global principal*)
For example, the extremal value property:

$$\text{local (pointwise) bounded (continuity)} + \text{compact} \rightsquigarrow \text{global bounded!}$$

¶ Definitions of various compactness.

We start with some definitions on coverings:

Definition 1.2. Let $(X, \mathscr{T})$ be a topological space, and $A \subset X$ be a subset.

- A family of subsets $\mathscr{U} = \{U_\alpha\}$ is called a *covering* of A if $A \subset \bigcup_\alpha U_\alpha$.
- A *covering* $\mathscr{U}$ is called a *finite covering* if it is a finite collection.
- A *covering* $\mathscr{U}$ is called an *open covering* if each U_α is open.
- A covering $\mathscr{V}$ is a *sub-covering* of $\mathscr{U}$ if $\mathscr{V} \subset \mathscr{U}$.
- A covering $\mathscr{V}$ is a *refinement* of $\mathscr{U}$ if for any $V \in \mathscr{V}$, there exists $U \in \mathscr{U}$ such that $V \subset U$.

One should be aware of the difference between a sub-covering and a refinement.

Now we extend the different aspects of $[0, 1]$ to different notions of compactness:[1]

Definition 1.3. Let $(X, \mathscr{T})$ be a topological space.

(1) We say X is *compact* in X if any open covering $\mathscr{U} = \{U_\alpha\}$ of X admits a finite sub-covering, i.e. there exists $\{U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_k}\} \subset \mathscr{U}$ s.t. $X = \bigcup_{i=1}^k U_{\alpha_i}$.[2]

(2) We say X is *sequentially compact* if any sequence $x_1, x_2 \cdots \in X$ admits a convergent subsequence $x_{n_1}, x_{n_2}, \cdots \to x_0 \in X$.

(3) We say X is *limit point compact* if for any infinite subset $S \subset X$, $S' \neq \emptyset$.

Remark 1.4. Suppose $A \subset X$ be a subset, then we say A is compact/sequentially compact/limit point compact if, when endowed with the subspace topology, $(A, \mathscr{T}_{subspace})$ is compact/sequentially compact/limit point compact. Note that by definition,

> $A \subset X$ is compact $\iff$ for any family of open sets $\mathcal{U} = \{U_\alpha\}$ in X satisfying $A \subset \bigcup_\alpha U_\alpha$, one can find $U_{\alpha_1}, \cdots, U_{\alpha_k} \in \mathcal{U}$ s.t. $A \subset \bigcup_{j=1}^k U_{\alpha_j}$.

¶ Examples of compactness.

Example 1.5. In the Euclidean space $\mathbb{R}^n$,

$$\text{bounded closed} \Leftrightarrow \text{compact} \Leftrightarrow \text{sequentially compact} \Leftrightarrow \text{limit point compact.}$$

Example 1.6. Consider the space $(X, \mathscr{T}_{\text{cofinite}})$.

- it is compact:

  Suppose $X \subset \bigcup_\alpha U_\alpha$. Choose arbitrary α_1. By definition, U_{α_1} is open, so its complement $X \setminus A_{\alpha_1}$ is a finite set. Now one only need to choose finitely many elements in $\mathscr{A}$ to cover it.
- It is also sequentially compact:

[1] We only extend three different aspects in the table. We will see in today's problem set that the fourth one, the nested sequence property, is equivalent to a new compactness: countably compact.

[2] The definition of compactness via covering property of open sets was first introduced by Alexandroff and Urysohn in 1924.

This is a consequence of Example 1.6 in Lecture 4: Consider any sequence $x_1, x_2, \cdots$. If no element repents infinitely many times, then the whole sequence converges to any point. If at least one element repent infinitely many times, then that gives us a "constant" subsequence which converges to that element itself.

- And it is limit point compact:
  Reason: For any infinite set S, $S' = X$ since $U \cap S \neq \emptyset$ for any open U.

Example 1.7. Consider $X = (\mathbb{N}, \mathscr{T}_{\text{discrete}}) \times (\mathbb{N}, \mathscr{T}_{\text{trivial}})$.

- It is NOT compact:
  Reason: Let $U_n = \{n\} \times \mathbb{N}$. Then $\{U_n\}_{n\in\mathbb{N}}$ is an open covering which has no finite sub-covering.
- It is also NOT sequentially compact.
  Reason: The sequence $\{x_n = (n, 1)\}$ has no convergent subsequence.
- BUT: it is limit point compact:
  Reason: In fact, for any $S \neq \emptyset$, we have $S' \neq \emptyset$, since if $(m_0, n_0) \in S$ and $n_1 \neq n_0$, then $(m_0, n_1) \in \{(m_0, n_0)\}' \subset S'$.

¶ Relations between various compactness.

It is not too hard to see that "limit point compact" is the weakest among the three:

Proposition 1.8. *Let X be any topological space.*

(1) If X is compact, then it is limit point compact.
(2) If X is sequentially compact, then it is limit point compact.

Proof. (1) Suppose X is compact, and $S \subset X$ is any subset. Suppose S has no limit point. Then S is closed since $S' = \emptyset \subset S$. For each point $a \in S$, since $a \notin S'$, one can find an open set $U_a \subset X$ s.t. $S \cap U_a = \{a\}$. Now $\{S^c, U_a \mid a \in S\}$ is an open covering of X. By compactness, there exists $a_1, \cdots, a_k \in S$ such that

$$X = S^c \cup (\bigcup_{i=1}^{k} U_{a_i}).$$

It follows that

$$S = S \bigcap X = (\bigcup_{i=1}^{k} U_{a_i}) \cap S = \{a_1, \cdots, a_k\}$$

is a finite subset.

(2) Suppose X is sequentially compact, and $S \subset X$ is any infinite set. Take any infinite sequence $\{x_1, x_2, \cdots\} \subset S$ s.t. $x_i \neq x_j$ for $i \neq j$. Then there exists a subsequence $x_{n_1}, x_{n_2}, \cdots \to x_0 \in X$. It follows from definition that

$$x_0 \in \{x_{n_1}, x_{n_2}, \cdots\}' \subset \{x_1, x_2, \cdots\}' \subset S'.$$

So $S' \neq \emptyset$. □

Remark 1.9. (1) We will see later: for topological spaces,
- “compact $\not\Leftarrow$ sequentially compact”,
- “compact $\not\Rightarrow$ sequentially compact”.

(2) We will prove: for metric spaces,

“compact $\Leftrightarrow$ sequentially compact $\Leftrightarrow$ limit point compact”.[3]

¶ Characterization of compactness via closed sets.

By applying the “open-closed duality”, we can convert “the definition of compact sets via open sets” to an equivalent definition via closed sets:

$X = \bigcup_\alpha U_\alpha$, U_α open $\Rightarrow \exists U_{\alpha_i}, X = \bigcup_{i=1}^k U_{\alpha_i}$.	$\Leftrightarrow$	$\emptyset = \bigcap_\alpha F_\alpha$, F_α closed $\Rightarrow \exists F_{\alpha_i}$, $\emptyset = \bigcap_{i=1}^k F_{\alpha_i}$.	$\Leftrightarrow$	$\bigcap_{i=1}^k F_{\alpha_i} \neq \emptyset$ for any finite collection $\{F_{\alpha_1}, \cdots, F_{\alpha_k}\}$ $\Rightarrow \bigcap_\alpha F_\alpha \neq \emptyset$.

So we arrive at

Proposition 1.10 (Characterize compactness via closed sets)**.**
A topological space X is compact if and only if it satisfies the following property:

> [**Finite Intersection Property**] *If $\mathscr{F} = \{F_\alpha\}$ is any collection of closed sets s.t. any finite intersection*
>
> $$F_{\alpha_1} \cap \cdots \cap F_{\alpha_k} \neq \emptyset,$$
>
> *then $\cap_\alpha F_\alpha \neq \emptyset$.*

As a consequence, we get

Corollary 1.11 (Nested sequence property)**.** *Let X be compact, and*

$$X \supset F_1 \supset F_2 \supset \cdots$$

be a nested sequence of non-empty closed sets. Then $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

¶ Characterization of compactness via basis/sub-basis.

It is NOT surprising that we can characterize compactness via “basis covering”:

Proposition 1.12. *Let $\mathcal{B}$ be a basis of $(X, \mathscr{T})$. Then X is compact if and only if any basis covering $\mathscr{U} \subset \mathcal{B}$ of X, one can find a finite sub-covering.*

Proof. Suppose X is compact, and let $\mathscr{U} \subset \mathcal{B}$ be any basis covering. Since $\mathcal{B} \subset \mathscr{T}$, $\mathscr{U}$ is automatically an open covering. So it admits a finite sub-covering.

Conversely suppose any basis covering of X admits a finite sub-covering, and let $\mathscr{U}$ be any open covering of X. For any $x \in X$, there exists $U^x \in \mathscr{U}$ and $U_x \in \mathcal{B}$ s.t.

$$x \in U_x \subset U^x.$$

[3] But they are not equivalent to “bounded closed” in a general metric space.

Since $\{U_x\}$ is a basis covering of X, there exist $U_{x_1}, \cdots, U_{x_m}$ s.t. $X = \bigcup_{i=1}^n U_{x_i}$. It follows that for $U^{x_1}, \cdots, U^{x_n} \in \mathscr{U}$, $X = \bigcup_{i=1}^n U^{x_i}$, so X is compact. □

It is natural to extend this proposition to sub-basis.

Theorem 1.13 (Alexander sub-basis theorem)**.** *Let $\mathcal{S}$ be a sub-basis of $(X, \mathscr{T})$. Then X is compact if and only if any sub-basis covering $\mathscr{U} \subset \mathcal{S}$ of X has a finite sub-covering.*

Surprisingly, the proof is much more harder and it is equivalent to the axiom of choice! We will postpone the proof to next lecture.

2. Proposition of compactness

¶ Compactness v.s. continuous map.

Among the three different compactness, compactness and sequentially compactness are more important because they are preserved under continuous maps:

Proposition 2.1. *Let $f : X \to Y$ be continuous.*

(1) If $A \subset X$ is compact, then $f(A)$ is compact in Y.
(2) If $A \subset X$ sequentially compact, then $f(A)$ is sequentially compact in Y.

Proof. (1) Suppose A is compact. Given any open covering $\mathscr{V} = \{V_\alpha\}$ of $f(A)$, the pre-image $\mathscr{U} = \{f^{-1}(V_\alpha)\}$ is an open covering of A. By compactness, there exists $\alpha_1, \cdots, \alpha_k$ such that $A \subset \bigcup_{i=1}^k f^{-1}(V_{\alpha_i})$. It follows $f(A) \subset \bigcup_{i=1}^k V_{\alpha_i}$, i.e. $f(A)$ is compact.

(2) For any sequence $y_1, y_2, \cdots$ in $f(A)$, there exists $x_1, x_2, \cdots$ in A such that $f(x_i) = y_i$. Since A is sequentially compact, there exists a convergent subsequence $x_{n_1}, x_{n_2}, \cdots \to x_0 \in A$. It follows from the continuity of f that $y_{n_1}, y_{n_2}, \cdots \to f(x_0) \in f(A)$. So $f(A)$ is sequentially compact. □

Remark 2.2. The image of a limit point compact space under a continuous map need not be limit point compact. For example, as we have just seen, the product space $X = (\mathbb{N}, \mathscr{T}_{\text{discrete}}) \times (\mathbb{N}, \mathscr{T}_{\text{trivial}})$ is limit point compact. We also know that the projection

$$\pi_1 : (\mathbb{N}, \mathscr{T}_{\text{discrete}}) \times (\mathbb{N}, \mathscr{T}_{\text{trivial}}) \to (\mathbb{N}, \mathscr{T}_{\text{discrete}})$$

is continuous. However, the image $(\mathbb{N}, \mathscr{T}_{\text{discrete}})$ is not limit point compact since $A' = \emptyset$ for any subset in any space with discrete topology.

Since a subset in $\mathbb{R}$ is compact (with respect to the usual topology) if and only if it is sequentially compact if and only if it is bounded and closed, we immediately get

Corollary 2.3 (The extremal value property)**.** *Let $f : X \to \mathbb{R}$ be any continuous map. If $A \subset X$ is compact or sequentially compact in X, then $f(A)$ is bounded in $\mathbb{R}$. Moreover, there exists $a_1, a_2 \in A$ s.t. $f(a_1) \le f(x) \le f(a_2)$ for $\forall x \in A$.*

Proof. By Proposition 2.1, $f(A)$ is bounded closed in $\mathbb{R}$. The conclusion follows. □

Since any quotient map is continuous, we have

Corollary 2.4. *The quotient space of any compact/sequentially compact space is still compact/sequentially compact.*

So in particular, $\mathbb{RP}^n$ and the Klein bottle are compact.

¶ Proper maps.

In general, the pre-image of a compact set under a continuous map is no longer compact. Such examples are easy to construct.

Definition 2.5. Let X, Y be topological spaces. A map $f : X \to Y$ is called a *proper* map if for any compact set $B \subset Y$, its pre-image $f^{-1}(B)$ is compact in X.

Remark 2.6. Why do we study proper maps? Here is one reason: Recall that the morphism between topological spaces are continuous maps, since they pull-back open sets to open sets. On the other hand, usually topological properties of compact sets are easier (since we have the "local-to-global" principle). As a result, some topological invariants are defined only for compact objects, or for "compactly-supported" objects. For the latter case, the correct "morphism" should be continuous proper maps, since proper maps can pull-back compactly-supported objects to compactly supported objects. This is the case, for example, in studying compactly supported cohomology groups.

¶ Subspace of a compact space.

As usual, we would like to construct new compact spaces from old compact spaces, or even non-compact spaces. The first candidates one can look at is: subspaces of a compact space. Unfortunately, it is easy to see that a compact space could have non-compact subspace, e.g. $(0, 1)$ is a subspace of $[0, 1]$.

We can take a closer look at the problem: which subsets of $[0, 1]$ remain to be compact? We know that a set in $\mathbb{R}$ is compact if and only if it is bounded and closed. If A is a subset of $[0, 1]$, it is automatically bounded. So for a subset $A \subset [0, 1]$ to be compact, it is enough to require A to be closed.

It turns out that for more general topological spaces, it is also enough to require closedness for a subset to be compact:

Proposition 2.7. *Let $A \subset X$ be a closed subset.*

(1) If X is compact, then A compact.
(2) If X is sequentially compact, then A is sequentially compact.
(3) If X is limit point compact, then A is limit point compact.

Proof. Let $A \subset X$ be closed.

(1) For any open covering $\mathscr{U}$ of A, $\mathscr{U} \cup \{A^c\}$ is an open covering of X, which admits a finite sub-covering $U_1, \cdots, U_m, A^c$. It follows

$$A \subset U_1 \cup \cdots \cup U_m,$$

i.e. $\{U_1, \cdots, U_m\}$ is a finite sub-covering of $\mathcal{U}$.

(2) For any sequence $x_1, x_2, \cdots \in A$, it is a sequence in X, and thus has a convergent sequence $x_{n_k} \to x_0 \in X$. Since A is closed and $x_0 \in A$.

(3) For any infinite subset $S \subset A$, we have $S' \neq \emptyset$ in X. But $S' \subset A' \subset A$, so $S' \neq \emptyset$ in A. □

¶ Compact v.s. Hausdorff.

However, it is important to point out that a compact subset of a compact set need not be closed. For example, for $(X, \mathscr{T}_{trivial})$, any subset is compact.

Definition 2.8. We say a topological space $(X, \mathscr{T})$ is *Hausdorff* if for any $x_1 \neq x_2 \in X$, there exists open sets $U_1 \ni x_1$ and $U_2 \ni x_2$ s.t. $U_1 \cap U_2 = \emptyset$

Remark 2.9. Hausdorff property is one of the most widely assumed property in applications. For example, one can easily see that (check!) in a Hausdorff space, if a sequence converges, then the limit is unique.

We will study Hausdorff property and other separation axioms in detail later.

Although it seems that compactness and Hausdorff property are very different, it turns out that they are "the dual" to each other in the following sense:

Proposition 2.10. *(1) If $(X, \mathscr{T})$ is compact, then*
- *(a) Every closed subset in X is compact.*
- *(b) If $\mathscr{T}' \subset \mathscr{T}$, then $(X, \mathscr{T}')$ is compact.*
- *(c) $(X, \mathscr{T}_{trivial})$ is always compact.*

(2) If $(X, \mathscr{T})$ is Hausdorff, then
- *(a) Every compact subset in X is closed.*
- *(b) If $\mathscr{T}' \supset \mathscr{T}$, then $(X, \mathscr{T}')$ is Hausdorff.*
- *(c) $(X, \mathscr{T}_{discrete})$ is always Hausdorff.*

Proof. We have proved (1)(a). It is trivial to check (1)(b), (1)(c) and (2)(b), (2)(c).

So it remains to prove (2)(a): Let $A \subset X$ be compact, $x_0 \in X \setminus A$. For any $y \in A$, we can find $U_y \ni x_0$ and $V_y \ni y$ s.t. $U_y \cap V_y = \emptyset$. Since $A \subset \cup_{y \in A} V_y$, one can find $y_1, \cdots, y_m$ s.t. $A \subset V_{y_1} \cup \cdots \cup V_{y_m} \Rightarrow U_{y_1} \cup \cdots \cup U_{y_m} \subset X \setminus A$. So $X \setminus A$ is open. □

Remark 2.11. So compact topologies "tends to be weak", Hausdorff topologies "tends to be strong". In particular, compact Hausdorff spaces form a very special class of topological spaces: If $\mathscr{T}$ is a compact Hausdorff topology on X, $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on X such that $\mathscr{T}_1 \subsetneqq \mathscr{T} \subsetneqq \mathscr{T}_2$, then $(X, \mathscr{T}_1)$ is NOT Hausdorff, and $(X, \mathscr{T}_2)$ is NOT compact. Thus for a compact Hausdorff space,

增之一分则太长，减之一分则太短；著粉则太白，施朱则太赤。