Topology (H) Lecture 9
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COMPACTNESS IN METRIC SPACE

1. Topological and non-topological aspects of metric spaces

¶ Some topological properties of all metrics spaces.

Let (X, d) be a metric space, with metric topology $\mathscr{T}_d$ generated by the basis

$$\mathcal{B} = \{B(x,r) \mid x \in X, r \in \mathbb{R}_{>0}\}.$$

Compared with general topological spaces, metric spaces have many nice properties:

(1) Any metric space is *first countable*, since at each $x \in X$, there is a countable neighborhood basis

$$\mathcal{B}_x = \{B(x,r) \mid r \in \mathbb{Q}_{>0}\}.$$

As a consequence, we get (c.f. Lecture 7, Prop. 1.10 and PSet 4-1-1)
- $F \subset X$ is closed if and only if it contains all its sequential limit points.
- A map $f : X \to Y$ is continuous if and only if it is sequentially continuous.

(2) Any metric space is *Hausdorff*, since for each $x \neq y \in X$, if we take $\delta = d(x,y)/2 > 0$, then

$$B(x,\delta) \cap B(y,\delta) = \emptyset.$$

As a consequence, we get
- every compact set in a metric space is closed.
- (in particular,) any single point set $\{x\}$ is closed.
- any convergent sequence in a metric space has a unique limit.[Prove this!]
  Note: This fact together with first countability implies: any sequentially compact set in X is closed. [Prove this!]

(3) In fact, in metric spaces, not only we can "separate" different points by disjoint open sets, but also we can "separate" disjoint closed sets by disjoint open sets: According to Urysohn's lemma in metric space (c.f. PSet 1-2-3(c)), for any pair of disjoint closed sets A and B in X, one can find a continuous function $f : X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B. As a consequence, the disjoint open sets $f^{-1}((-\infty, 1/3))$ and $f^{-1}((2/3, +\infty))$ separates the closed sets A and B. Such a topological property will be called *normal* and will be studied in more details later in this course.

Remark 1.1. In this course we also study other topological properties, e.g. compactness, second countable, connectedness etc. Most of these properties are only satisfied by some metric spaces.

¶ Metric aspects of metric spaces: boundedness.

In Lecture 2 we defined *diameter* (and thus the conception of *boundedness*) of a subset in a metric space (X, d):

$$diam(A) = \sup\{d(x, y) \mid x, y \in A\} \quad (\leq +\infty).$$

We have seen at the beginning of Lecture 3 that diameter and boundedness are not a topological conceptions: If you change the metric to a topologically equivalent metric, the diameter could change, and a bounded set could become unbounded. However, it is easy to see that any unbounded set is neither compact nor sequentially compact. So

$$\boxed{\text{Any compact/sequentially compact subset in } (X, d) \text{ is bounded and closed.}}$$

Conversely, it is easy to find bounded and closed subsets that are not compact:

Example 1.2. (1) $(\mathbb{N}, d_{\text{discrete}})$ is bounded and closed in $(\mathbb{N}, d_{\text{discrete}})$.
(2) $(\mathbb{R}, \frac{d}{d+1})$ is a bounded and closed in $(\mathbb{R}, \frac{d}{d+1})$.
(3) $((0, 1], d_{Euclidian})$ is bounded and closed in $((0, +\infty), d_{Euclidian})$.

As a consequence, "closed and bounded" will not be an equivalent condition to characterize compactness in metrics spaces.

¶ Metric aspects of metric spaces: totally boundedness.

After carefully studying (1) and (2) in Example 1.2 above, you will find that they are "bad" bounded spaces: we can cover them by one large ball, say, a ball of radius 2; but we can't cover them by finite many balls of small radius, say, balls of radius $\frac{1}{2}$. For example, in case (2), each interval $(n, n+1)$ has length $\frac{1}{2}$ with respect to the metric $d/(d+1)$. In other words, these "bad" spaces are bounded when you measure them with a long ruler, but they are not bounded when you measure them with short rulers!

Definition 1.3. We say a metric space (X, d) is *totally-bounded* if for any $\varepsilon > 0$, we can cover X by finitely many balls of radius ε.

Obviously any totally bounded space is bounded, but the converse is not true.

Remark 1.4. By definition, a metric space (X, d) is totally bounded if and only if for any $\varepsilon > 0$, there exists a finite set $\{x_1, \cdots, x_{n(\varepsilon)}\}$ such that

$$\forall y \in X, \quad \text{there exists } 1 \leq i \leq n(\varepsilon) \text{ s.t. } d(x_i, y) < \varepsilon.$$

Definition 1.5. (a) A set of points N in a metric space (X, d) is called an *ε-net* if

$$\forall y \in X, \quad \text{there exists } x \in N \text{ s.t. } d(x, y) < \varepsilon.$$

(b) A ε-net is called a *finite ε-net* if it is a finite set.

So by definition,

A metric space is totally bounded $\iff$ it admits a finite ε-net for any $\varepsilon > 0$.

It turns out that for a metric space, compactness implies totally boundedness:

Proposition 1.6. *If (X, d) is compact/sequentially compact, then it is totally bounded.*

Proof. If (X, d) is compact, then it is totally bounded, since the open covering $\{B(x, \varepsilon) \mid x \in X\}$ has a finite sub-covering.

Now suppose (X, d) is sequentially compact and suppose on the contrary that there exists $\varepsilon > 0$ s.t. X can't be covered by finitely many ε-balls. Take any $x_1 \in X$. Since $X \setminus B(x_1, \varepsilon) \neq \emptyset$, one can pick $x_2 \in X \setminus B(x_1, \varepsilon)$. Inductively we can find $x_1, x_2, \cdots$ s.t.

$$X \setminus \cup_{i=1}^n B(x_i, \varepsilon) \neq \emptyset, \quad \forall\, n.$$

Now we get a sequence $\{x_n\}$ with

$$d(x_n, x_m) > \varepsilon, \quad \forall\, n \neq m.$$

So $\{x_n\}$ has no convergent subsequence. A contradiction. $\square$

¶ Metric aspects of metric spaces: Lebesgue number lemma.

Another very useful metric property is the so-called Lebesgue number lemma. We have seen the lemma for Euclidian spaces. Now we generalize it to metric spaces:

Proposition 1.7 (Lebesgue Number Lemma)**.** *If (X, d) is sequentially compact, then for any open covering $\mathscr{U}$ of X, there exists $\delta > 0$ (which depends on $\mathscr{U}$)*[1] *such that for any subset $A \subset X$ with $diam(A) < \delta$, there exists $U \in \mathscr{U}$ s.t. $A \subset U$.*

Proof. By contradiction. Let $\mathscr{U}$ be an open covering of X such that for any $n \in \mathbb{N}$, there exists $C_n \subset X$ with $diam(C_n) < \frac{1}{n}$ s.t. C_n is NOT contained in any $U \in \mathscr{U}$. We pick $x_n \in C_n$. Then $\{x_n\}$ is a sequence in X. Since (X, d) is sequentially compact, there exists a subsequence $x_{n_k} \to x_0 \in X$. Since $\mathscr{U}$ is an open covering of X, one can find $U \in \mathscr{U}$ s.t. $x_0 \in U$. Now we choose $\varepsilon_0 > 0$ s.t. $B(x_0, \varepsilon_0) \subset U$, and then choose n_k such that

$$\frac{1}{n_k} < \frac{\varepsilon_0}{2} \quad \text{and} \quad d(x_{n_k}, x_0) < \frac{\varepsilon_0}{2}.$$

If follows that

$$C_{n_k} \subset B(x_{n_k}, \frac{1}{n_k}) \subset B(x_0, \varepsilon_0) \subset U,$$

a contradiction! $\square$

[1]Such a number δ is called a *Lebesgue number* of the cover $\mathscr{U}$.

Remark 1.8. Let $\gamma : [0,1] \to X$ be a continuous map, and $X = \bigcup_\alpha U_\alpha$ is an open covering of X. Then $\gamma^{-1}(U_\alpha)$ is an open covering of $[0,1]$. By Lebesgue number lemma, there exists $\delta > 0$ such that each interval $[t, t+\delta]$ is contained in some $\gamma^{-1}(U_\alpha)$. This simple fact will play an important role in the study of fundamental groups in the second part of this course.

¶ Metric aspects of metric spaces: completeness.

Another very useful metric but non-topological conception for metric spaces is *completeness*. We need

Definition 1.9. A sequence $\{x_n\}$ in a metric space (X,d) is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists $N > 0$ such that

$$d(x_n, x_m) < \varepsilon, \quad \forall n, m > N.$$

As in the Euclidean case, one can easily prove

$$\boxed{\text{If } x_n \to x_0 \text{ in } (X,d), \text{ then } \{x_n\} \text{ is a Cauchy sequence in } (X,d).}$$

Conversely it is possible that a Cauchy sequence does not converge.

Definition 1.10. We say (X,d) is *complete* if any Cauchy sequence in X converges.

Example 1.11. $\mathbb{R}$ and $[0,1]$ are complete w.r.t. $d_{\text{Euclidean}}$, while $\mathbb{Q}$ and $(0,1)$ are not. (1) and (2) in Example 1.2 are complete while (3) is not.

Example 1.12. Most interesting metric spaces in analysis, including Banach spaces, Hilbert spaces, Frechet spaces etc, are (required to be) complete.

Example 1.13. Since any closed set in a metric space contains all its sequential limit points, and since any Cauchy sequence in a metric subspace is automatically a Cauchy sequence in the original space, we conclude

$$\boxed{\text{If } (X,d) \text{ is complete, and } F \subset X \text{ is closed, then } (F,d) \text{ is complete.}}$$

Example 1.14. Suppose (X,d) is a sequentially compact metric space. Given any Cauchy sequence $\{x_n\}$ in X, by sequentially compactness, one can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to $x_0 \in X$. Then use the definition of Cauchy sequence and the triangle inequality, one can easily prove $x_n \to x_0$. Thus we conclude

$$\boxed{\text{Any sequentially compact metric space is complete.}}$$

¶ Detour: completion.

Example 1.15. Given any set X and any complete metric space (Y, d_Y), the space of bounded maps,

$$\mathcal{B}(X,Y) = \{f : X \to Y \mid f(X) \text{ is bounded in } Y\}$$

is complete with respect to the supremum metric $d_\infty(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}$. Details left as an exercise.

Recall (lecture 2) that a map $f : (X, d_X) \to (Y, d_Y)$ is an *isometric embedding* if

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2), \ \forall x_1, x_2 \in X.$$

Definition 1.16. We say a complete metric space $(\widetilde{X}, \tilde{d})$ is a *completion* of a metric space (X, d) if there exists an isometric embedding $f : X \to \widetilde{X}$ such that $\overline{f(X)} = \widetilde{X}$.

It turns out that (details left as an exercise) any metric space (X, d_X) can be isometrically embedded into $(\mathcal{B}(X, \mathbb{R}), d_\infty)$, and as a consequence,

Theorem 1.17. *Any metric space (X, d) admits a completion $(\widetilde{X}, \tilde{d})$ (which is unique up to isometry).*

¶ Detour: complete = absolutely closed.

Now we take a closer look at (3) in Example 1.2. The metric space $((0, 1], d)$ is totally bounded, and $(0, 1]$ is closed in $(0, +\infty)$. However, if we embed $(0, 1]$ isometrically into a different metric space, say $\mathbb{R}$, then $(0, 1]$ is no longer closed. We will call a metric space *absolutely closed* if it satisfies the following stronger closedness condition:

(AC) If (Y, d) is any metric space, and $f : (X, d_0) \to (Y, d)$ is an isometric embedding, then $f(X)$ is closed in Y.

It turns out that absolutely closedness is NOT a new conception:

Proposition 1.18. *A metric space is absolutely closed if and only if it is complete.*

Proof. If (X, d_0) satisfies (AC), and $\{x_n\}$ is a Cauchy sequence in (X, d). Then by embedding (X, d_0) into its completion $(\widetilde{X}, \tilde{d})$ and identifying each $x \in X$ with its image, we conclude that $\{x_n\}$ is a Cauchy sequence in $(\widetilde{X}, \tilde{d})$. Since $(\widetilde{X}, \tilde{d})$ is complete, x_n converges to a unique $\tilde{x} \in (\widetilde{X}, \tilde{d})$. Since (X, d) is closed in $(\widetilde{X}, \tilde{d})$, we conclude that $\tilde{x} \in X$. So (X, d) is complete.

Conversely suppose (X, d) is complete and (X, d) can be isometrically embedded int (Y, d_Y). Then as a subset of (Y, d_Y), X contains all its sequential limit points and thus is closed in (Y, d_Y). □

So we get a new explanation of completeness of metric spaces:

$$\boxed{\text{“complete”} = \text{“always closed as subspace”}.}$$

This also give a different proof of the fact that any compact/sequentially compact metric space is complete: since

- any isometric embedding must be continuous,
- the image of a compact/sequentially compact set under a continuous map is compact/sequentially compact,
- in a metric space is compact, any compact/sequentially compact set is closed.

We conclude that any compact/sequentially compact metric space is absolutely closed, i.e. is complete.

2. Equivalence of various compactness for metric spaces

¶ limit point compact $\Longleftrightarrow$ sequentially compact in metric space.

We have seen that for any topological space,

$$\text{compact} \implies \text{limit point compact} \impliedby \text{sequentially compact}$$

Our first observation is

Proposition 2.1. *For metric spaces, limit point compact $\Longleftrightarrow$ sequentially compact.*

Proof. Only need to show limit point compact $\Longrightarrow$ sequentially compact for (X, d).

Let $\{x_n\}$ be any sequence in (X, d). If the set $A = \{x_n \mid n \in \mathbb{N}\}$ is a finite set, then by the pigeonhole principle, $\{x_n\}$ has a constant subsequence

$$x_{n_1} = x_{n_2} = \cdots = x_0,$$

which is the convergent subsequence we are looking for.

Now suppose the set $A = \{x_n \mid n \in \mathbb{N}\}$ is a infinite set, then by limit point compactness, $A' \neq \emptyset$. Take any $x_0 \in A'$. By definition, for any $k \in \mathbb{N}$, we have

$$B(x_0, 1/k) \cap (A \setminus \{x_0\}) \neq \emptyset.$$

We claim that in fact, each $B(x_0, 1/k) \cap (A \setminus \{x_0\})$ is an infinite set. Otherwise, if

$$B(x_0, 1/k) \cap (A \setminus \{x_0\}) = \{x_{m_1}, \cdots, x_{m_k}\}$$

for some k, then we take N large s.t. $1/N < \min(d(x_0, x_{m_k}))$. It follows that

$$B(x_0, 1/N) \cap (A \setminus \{x_0\}) = \emptyset,$$

which is a contradiction. So we can find $n_1 < n_2 < \cdots$ s.t. $x_{n_k} \in B(x_0, 1/k)$. Obviously the subsequence $x_{n_k} \to x_0$. □

Remark 2.2. In the proof, we only used the (A1) and (T2) properties of a metric space. [where? find out them!] Modifying the proof slightly, one can prove

Proposition 2.3. *If X is a topological space which is Hausdorff and first countable, then in X we have "limit point compact" $\Longleftrightarrow$ "sequentially compact".*

¶ sequentially compact $\Longleftrightarrow$ "totally bounded and absolutely closed".

Now we prove the correct generalization of "compact $\Longleftrightarrow$ closed and bounded in $\mathbb{R}^m$": For a general metric space, we need to replace "closed" by "absolutely closed" and replace "bounded" by "totally bounded", namely

Proposition 2.4. *A metric space (X, d) is sequentially compact if and only if it is complete and totally bounded.*

Proof. We have proven that any sequentially compact metric space is complete and totally bounded. Now suppose (X, d) is complete and totally bounded, and let $\{x_n\} \subset X$ be a sequence. Since X is totally bounded, we can cover X by finitely many radius-1 balls. Then there exists a ball B_1 in this finite covering such that

$$J_1 := \{n \in \mathbb{N} \mid x_n \in B_1\}$$

is an infinite set. Then we cover X by finitely many radius-$\frac{1}{2}$ balls. Again there exists a ball B_2 in this new finite covering such that

$$J_2 := \{n \in J_1 \mid x_n \in B_2\}$$

is an infinite set. Continuing this construction, we get a sequence of indices,

$$\mathbb{N} \supset J_1 \supset J_2 \supset \cdots$$

s.t. each J_k is an infinite set, and

$$i, j \in J_k \implies d(x_i, x_j) < \frac{2}{k}.$$

Now we take $n_i \in J_i$ such that $n_1 < n_2 < \cdots$. Then $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$, and is a Cauchy sequence. Since (X, d) is complete, we conclude that $\{x_{n_i}\}$ converges to a point $x_0 \in X$. So (X, d) is sequentially compact. □

¶ Equivalence of different characterizations of compactness in metric space.

Putting all "jigsaw" pieces together, we conclude:

Theorem 2.5 (Compactness in metric space)**.** *In a metric space (X, d), the following "compactness" are all equivalent:*

(1) A is compact.
(2) A is sequentially compact.
(3) A is limit point compact.
(4) A is complete and totally bounded.

Proof. We have seen (1) $\implies$ (3) $\iff$ (2) $\iff$ (4). It remains to show (2) $\implies$ (1) for metric space.

Suppose $A \subset (X, d)$ is sequentially compact, and let $\mathscr{U}$ be any open covering. On one hand, by Lebesgue Number Lemma, there exists a Lebesgue number $\delta > 0$ such that any set of radius less than δ can be covered by an open set in $\mathscr{U}$. On the other hand, by Proposition 1.6, X is totally bounded and thus can be covered by finitely many $\delta/2$-balls. It follows that $\mathscr{U}$ has a finite sub-covering. □

Remark 2.6. Obviously we can add a 5^{th} equivalent characterization to the theorem:

(5) A is countably compact.

We will see that there is a 6^{th} equivalent characterization:

(6) A is pseudo-compact, namely any continuous function $f : A \to \mathbb{R}$ is bounded.

¶ A "constructive" proof of Lebesgue Number Lemma.

The proof of the Proposition 1.7 (Lebesgue number lemma) is based on contradiction argument and thus is non-constructive. In particular, given an open covering $\mathscr{U}$, that proof does not tell us how large can a Lebesgue number δ be. Now we can give a more "constructive" proof which tells us how large a Lebesgue number δ can be: [2]

A "constructive" proof of Lebesgue Number Lemma. .
Let $\mathscr{U}$ be an open covering of a compact metric space (X, d). By compactness, $\mathscr{U}$ has a finite sub-covering $\{U_1, \cdots, U_n\}$. We may assume each $U_k \neq X$, otherwise we can take δ to be any number. Let $F_k = U_k^c$. Then each F_k is a closed set in the compact space, and thus are compact. We define a function $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{k=1}^{n} d(x, F_k).$$

Then f is continuous, and $f(x) > 0$ everywhere since any $x \in X$ is contained in some open set U_k. It follows from the compactness of X and the extremal value theorem that f attains a positive minimum on X,

$$\forall x \in X, \quad f(x) \geq f(x_0) =: \delta > 0.$$

It is not hard to verify that such a δ is a Lebesgue number: For any subset $A \subset X$ with $\operatorname{diam}(A) < \delta$, we choose any $x_1 \in A$. Then $A \subset B(x_1, \delta)$. Since $f(x_1) \geq \delta$, there must exists at least one k such that $d(x_1, F_k) \geq \delta$. It follows that $B(x_1, \delta) \subset U_k$, and thus $F \subset U_k$. □

[2] In mathematics, especially in analysis, usually "quantitative" proofs will give you more information than "qualitative" proofs.