Topology (H) Lecture 11
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STONE-WEIERSTRASS THEOREM

1. The uniform topology

¶ Three Topologies on $\mathcal{M}(X, Y)$.

Let X be any set, and Y a metric space. Consider the space of maps,

$$\mathcal{M}(X,Y) = Y^X = \text{ the space of all maps from } X \text{ to } Y.$$

On $\mathcal{M}(X,Y) = Y^X$ we have studied three topologies:

(1) the product topology, generated by the sub-basis

$$\mathcal{S}_{product} = \left\{ \pi_x^{-1}\big(B^Y(y_x, r_x)\big) \;\middle|\; \forall x \in X, \forall y_x \in Y, \forall r_x > 0 \right\}.$$

(2) the box topology, generated by the basis

$$\mathcal{B}_{box} = \left\{ \prod_{x\in X} \big(B^Y(y_x, r_x)\big) \;\middle|\; \forall y_x \in Y, \forall r_x > 0 \right\}.$$

(3) Since Y is a metric space, we can define the *uniform metric* (See PSet 1-2-4-c)

$$d_u(f,g) := \sup_{x\in X} \frac{d_Y\big(f(x), g(x)\big)}{1 + d_Y\big(f(x), g(x)\big)}.$$

We have seen in PSet 1-2-4-c that d_u is a metric on $\mathcal{M}(X,Y)$ and

$$f_n \to f \text{ uniformly }^1 \iff f_n \to f \text{ in } (\mathcal{M}(X,Y), d_u).$$

Definition 1.1. The metric topology induced by d_u on $\mathcal{M}(X,Y)$ is called the *uniform topology* on $\mathcal{M}(X,Y)$.

Remarks 1.2. As we know, the product topology coincides with the "pointwise convergence topology". Just as PSet 3-1-3(a), one can prove: the uniform topology is weaker than the box topology, but stronger than the product topology. Moreover, for any infinite set X and "non-trivial" Y, the three topologies are pairwise different. [They are the same for finite set X.]

Similar to the proof of completeness of $\mathcal{B}(X,Y)$ in PSet 5-1-1(b), we have (exercise.)

Proposition 1.3. *Suppose Y is complete. Then d_u is a complete metric on $\mathcal{M}(X,Y)$.*

[1]Note: To define "uniform convergence" for maps in $\mathcal{M}(X,Y)$, we don't need metric structure or even topological structure on X. We only need the metric structure on Y.

¶ The uniform topology on $\mathcal{C}(X,Y)$.

Now suppose $(X,\mathscr{T})$ is a topological space. Then we can talk about continuity of maps in $\mathcal{M}(X,Y)$. In particular, we can study the space of continuous maps,

$$\mathcal{C}(X,Y) := \{f \in \mathcal{M}(X,Y) \mid f \text{ is continuous }\}.$$

As in PSet 1-2-4(b), we have (exercise)

Proposition 1.4. *$\mathcal{C}(X,Y)$ is a closed subset of $(\mathcal{M}(X,Y),d_u)$.*

Remark 1.5. In general $\mathcal{C}(X,Y)$ is NOT closed in $(\mathcal{M}(X,Y),\mathscr{T}_{product})$ or $(\mathcal{M}(X,Y),\mathscr{T}_{box})$.

As a consequence,

Corollary 1.6. *If Y is complete, then $(\mathcal{C}(X,Y),d_u)$ is complete.*

Remark 1.7. Suppose X is compact. Then $\mathcal{C}(X,Y)\subset\mathcal{B}(X,Y)$. On $\mathcal{B}(X,Y)$ we have a simpler metric $d_\infty(f,g) := \sup_{x\in X}|f(x)-g(x)|$. It is easy to prove that $f_n\to f$ with respect to d_u if and only if $f_n\to f$ with respect to d_∞. In other words, both d_u and d_∞ induces the same topology on $\mathcal{C}(X,Y)$. So in the case X is compact, we may (and will) use d_∞ instead of d_u, to make computation a bit simpler.

In the remaining of today's lecture, we will concentrate only on $\mathcal{C}(X,\mathbb{R})$ or $\mathcal{C}(X,\mathbb{C})$, viewed as a subspace in $\mathcal{M}(X,\mathbb{R})$ or $\mathcal{M}(X,\mathbb{C})$. We will assume X is compact and thus use the metric d_∞.

2. The Stone-Weierstrass Theorem

¶ Weierstrass Approximation Theorem.

We are familiar with the space $(\mathcal{C}([0,1],\mathbb{R}),d_\infty)$ in mathematical analysis. In particular, we have learned

Theorem 2.1 (Weierstrass Approximation Theorem, 1885)**.** *The set of polynomials, $\mathcal{P}([0,1])$, is dense in $(\mathcal{C}([0,1],\mathbb{R}),d_\infty)$. In other words, for any $\varepsilon>0$ and any $f\in\mathcal{C}([0,1],\mathbb{R})$, there exists a polynomial P such that*

$$\sup_{x\in[0,1]}|f(x)-P(x)|<\varepsilon.$$

Proof. Here is a "probabilistic proof" by S. Bernstein[2] in 1912:

$$B_n(f)(x) := \sum_{i=0}^n f(\frac{i}{n})\cdot\binom{n}{i}x^i(1-x)^{n-i} \xrightarrow{\text{uniformly}} f.$$ ∎

As a consequence,

[2]Sergei Natanovich Bernstein, 1880-1968, Russian and Soviet mathematician known for contributions to partial differential equations, differential geometry, probability theory, and approximation theory. He was the founder of the constructive theory of functions, and he introduced a priori estimates in PDE.

Corollary 2.2. *For any $0 < \varepsilon, \delta < \frac{1}{2}$, there exists a polynomial $q = q(t)$ with $q(0) = 0$ and $q([0,1]) \subset (-\delta, 1+\delta)$, such that*

$$q([0,\varepsilon)) \subset (-\delta, \delta) \quad \textit{and} \quad q((2\varepsilon, 1]) \subset (1-\delta, 1+\delta).$$

Proof. According to the Weierstrass approximation theorem, there is a polynomial $q_1 \in \mathcal{P}([0,1])$ such that:

$$|q_1(t) - f_0(t)| < \frac{\delta}{2}, \qquad \text{where} \quad f_0(t) = \begin{cases} 0 & \text{on } [0,\varepsilon], \\ \text{linear} & \text{on } [\varepsilon, 2\varepsilon], \\ 1 & \text{on } [2\varepsilon, 1]. \end{cases}$$

Then let $q(t) = q_1(t) - q_1(0)$. [3] □

¶ $\mathcal{C}(X, \mathbb{R})$ as a unitary algebra.

One main purpose today is to extend Weierstrass approximation theorem to more general topological spaces. In the remaining of this lecture, unless otherwise stated we will always assume

$$\boxed{\text{Assumption: } X \text{ is a compact Hausdorff space.}}$$

Of course in general we will no longer have the conception of polynomials on topological spaces. But still we can ask:

$\boxed{\textbf{Question:}}$ Can we approximate $\mathcal{C}(X, \mathbb{R})$ by a relative simple subset of functions?

In the case of Weierstrass approximation theorem, we used the subset

$$\mathcal{P}([0,1]) = \text{ the space of polynomials.}$$

Observation: $\mathcal{C}([0,1], \mathbb{R})$ is an algebra, and $\mathcal{P}([0,1])$ is a subalgebra in $\mathcal{C}([0,1], \mathbb{R})$.

Definition 2.3. An *algebra* $\mathcal{A}$ is a vector space with a bilinear multiplication structure which is distributive. In other words, an algebra is a triple $(\mathcal{A}, +, \cdot)$ such that

- $(\mathcal{A}, +)$ is a vector space (over a field, say, $\mathbb{R}$ or $\mathbb{C}$).
- The product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a binary operation such that for any $x, y, z \in \mathcal{A}$ and any scalars a, b,
  - (distributive) $(x+y)\cdot z = x \cdot z + y \cdot z, \quad x \cdot (y+z) = x \cdot y + x \cdot z$.
  - (compatibility) $(ax) \cdot (by) = (ab)(x \cdot y)$.

Definition 2.4. An algebra is *unitary* (or *unital*) if it has an identity element with respect to the multiplication: $1 \cdot x = x \cdot 1 = x$.

Obviously both $\mathcal{C}([0,1], \mathbb{R})$ and $\mathcal{P}([0,1])$ are unitary algebras.

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[3] One can also construct a function of the form $q(t) = 1 - (1 - t^m)^n$ explicitly.

It is trivial to define the conception of subalgebra. As in the case of topological vector space, one can also define a *topological algebra* to be a topological vector space $\mathcal{A}$ which is also an algebra, such that the product map $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is (jointly) continuous. A subalgebra of a topological algebra is called a *closed subalgebra* if it is both a subalgebra and a closed subspace. One can prove

Proposition 2.5. *Let $\mathcal{A}$ be a topological algebra, and $\mathcal{A}_1 \subset \mathcal{A}$ a subalgebra. Then the closure $\overline{\mathcal{A}_1}$ is a (closed) subalgebra of $\mathcal{A}$.*

For the remaining of this section, we will endow $\mathcal{C}(X, \mathbb{R})$ with the uniform topology (the topology generated by d_∞), so that $\mathcal{C}(X, \mathbb{R})$ is a unitary topological algebra.

¶ Two conditions: "vanishes at no point" and "separates points".

Now let $\mathcal{A} \subset \mathcal{C}(X, \mathbb{R})$ be a subalgebra. We want to find out conditions such that $\mathcal{A}$ is dense in $\mathcal{C}(X, \mathbb{R})$. To do so, we first start with examples where $\mathcal{A}$ is NOT dense:

Example 2.6.

(1) Consider
$$\mathcal{A} = \left\{ f = \sum_{k=1}^n a_k x^k \;\middle|\; n \in \mathbb{N}, a_k \in \mathbb{R} \right\} \subset \mathcal{C}([0,1], \mathbb{R}).$$
Then $\mathcal{A}$ is a sub-algebra in $\mathcal{C}([0,1], \mathbb{R})$ but it is NOT dense since
$$f(0) = 0, \qquad \forall f \in \mathcal{A},$$
which implies: by using functions in $\mathcal{A}$ only, you can't approximate any function that is nonzero at $x = 0$.

(2) Consider
$$\mathcal{A} = \left\{ f = \sum_{k=0}^n (a_k \cos(kx) + b_k \sin(kx)) \;\middle|\; n \in \mathbb{N}, a_k, b_k \in \mathbb{R} \right\} \subset \mathcal{C}([0, 2\pi], \mathbb{R}).$$
Then $\mathcal{A}$ is a subalgebra of $\mathcal{C}([0, 2\pi], \mathbb{R})$ but it is NOT dense since
$$f(0) = f(2\pi), \qquad \forall f \in \mathcal{A},$$
which implies: by using functions in $\mathcal{A}$ only, you can't approximate any function with $f(0) \neq f(2\pi)$.

It turns out that they are "the only bad examples".

Definition 2.7. We say a subalgebra $\mathcal{A} \subset \mathcal{C}(X, \mathbb{R})$

(1) *vanishes at no point* if $\forall x \in X, \exists f \in \mathcal{A}$ such that $f(x) \neq 0$.
(2) *separates points* if $\forall x \neq y \in X, \exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.[4]

Obviously if $\mathcal{A}$ is unitary, then it vanishes at no point. Conversely,

[4]Note: X has to be Hausdorff, otherwise no subalgebra of $\mathcal{C}(X, \mathbb{R})$ separates points.

Proposition 2.8. *If a subalgebra $\mathcal{A}$ of $\mathcal{C}(X,\mathbb{R})$ vanishes at no point, then $1 \in \overline{\mathcal{A}}$, i.e. $\overline{\mathcal{A}}$ is unitary.*

Proof. For any $x \in X$, there exists $f_x \in \mathcal{A}$ such that $f_x(x) \neq 0$. Let

$$U_x = \{y \mid f_x(y) \neq 0\}.$$

Then $\{U_x\}$ is an open covering of X. So there exist points $x_1, \cdots, x_m$ such that $X \subset U_{x_1} \cup \cdots \cup U_{x_m}$. Let

$$f_1(x) = f_{x_1}^2 + \cdots + f_{x_m}^2 \in \mathcal{A}.$$

Then $f_1(x) > 0$ for all $x \in X$. By compactness of X, there exist $a, b > 0$ such that $a \leqslant f_1(x) \leqslant b$ for any $x \in X$. This shows that

$$\frac{a}{b} \leqslant \frac{f_1(x)}{b} \leqslant 1, \forall x \in X.$$

For any $\delta > 0$, by Corollary 2.2, there exists $q \in \mathcal{P}([0,1])$ with $q(0) = 0$ such that

$$f(x) := q(\frac{f_1(x)}{b}) \subset (1-\delta, 1+\delta),$$

i.e. $d_\infty(f, 1) < \delta$. Finally $f \in \mathcal{A}$ since $q \in \mathcal{P}([0,1])$ with $q(0) = 0$. □

Note: In the proof we actually proved: A subalgebra of $\mathcal{C}(X,\mathbb{R})$ vanishes at no point if and only if it contains a nonzero function.

¶ Stone-Weierstrass Theorem, Version 1.

In 1937, M. Stone[5] generalized Weierstrass approximation theorem to compact Hausdorff spaces:

Theorem 2.9 (Stone-Weierstrass Theorem for compact Hausdorff space, Version 1)**.** *Let X be any compact Hausdorff space. Let $\mathcal{A} \subset \mathcal{C}(X,\mathbb{R})$ be a subalgebra which vanishes at no point <u>and</u> separates points. Then $\mathcal{A}$ is dense in $\mathcal{C}(X,\mathbb{R})$.*

Stone-Weierstrass theorem is one of the most useful theorem in modern analysis. Its importance cannot be overemphasized. In his book "General Topology" Kelly wrote

> This is unquestionably the most useful known result on $\mathcal{C}(X)$.

[5]Marshall Harvey Stone (1903-1989), American mathematician who contributed to real analysis, functional analysis, topology and the study of Boolean algebras. He is known for *Stone-von Neumann theorem* (1930), *Stone-Čech compactification* (1934), *Stone's representation theorem* and *Stone duality* (1936), *Banach-Stone theorem* (1937), *Stone-Weierstrass Theorem* (1937 and 1948) etc. NOTE: There was another mathematician "Stone" who contributed to topology, Arthur Harold Stone (1916-2000), a British mathematician. A. Stone is known for *Stone's theorem* (Every metric space is paracompact) and *Stone metrization theorem*.

We will see some of its applications in today's PSet. The theorem has been further generalized or extended to many different contexts, and there still exist attractive unsolved problems associated with it.

For the remaining of today's lecture we will prove Stone-Weierstrass theorem, and discuss several of its generalizations. You can see how such a great theorem grows in some expected and some unexpected way.

¶ Stone-Weierstrass Theorem, Version 2 and proof.

According to Proposition 2.5 and Proposition 2.8, Theorem 2.9 is equivalent to

Theorem 2.10 (Stone-Weierstrass Theorem for compact Hausdorff space, Version 2)**.** *Let X be compact Hausdorff, and $\mathcal{A} \subset \mathcal{C}(X,\mathbb{R})$ be a closed unitary subalgebra which separates points. Then $\mathcal{A} = \mathcal{C}(X,\mathbb{R})$.*

Proof. [6] We start with two observations which hold for any closed unitary subalgebra:

Observation 1. $f \in \mathcal{A} \Longrightarrow |f| \in \mathcal{A}$.

> Reason: Since f is bounded, by Weierstrass approximation theorem, there exists a sequence of polynomials $p_n(t) \to \sqrt{t}$ uniformly on $[0, |f|_\infty^2]$. So
> $$p_n \circ f^2 \longrightarrow \sqrt{f^2} = |f| \quad \text{uniformly.}$$
> By closedness of $\mathcal{A}$, $|f| \in \mathcal{A}$.

Observation 2. $f_k \in \mathcal{A} \Longrightarrow \max\{f_1, \cdots, f_n\} \in \mathcal{A}$, $\min\{f_1, \cdots, f_n\} \in \mathcal{A}$.

> This follows from
> $$\max\{f,g\} = \frac{f+g+|f-g|}{2}, \quad \min\{f,g\} = \frac{f+g-|f-g|}{2}.$$

Now we prove the theorem. Let $f \in \mathcal{C}(X,\mathbb{R})$. We need to find $f_\varepsilon \in \mathcal{A}$ such that $d_\infty(f, f_\varepsilon) < \varepsilon$. For any $a \neq b \in X$, since $\mathcal{A}$ separates points, there exists $g \in \mathcal{A}$ such that $g(a) \neq g(b)$. Let

$$f_{a,b}(x) = f(a) + \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)).$$

Then $f_{a,b} \in \mathcal{A}$ and $f_{a,b}(a) = f(a)$, $f_{a,b}(b) = f(b)$. Now consider the sets

$$U_{a,b,\varepsilon} := \{x \in X \mid f_{a,b}(x) < f(x) + \varepsilon\}.$$

By continuity of f and $f_{a,b}$, it is open. Moreover, for any b and ε fixed, $\{U_{a,b,\varepsilon}\}_{a \in X}$ is an open covering of X. By compactness of X, we can find a finite sub-covering

$$\{U_{a_1(b,\varepsilon),b,\varepsilon}, U_{a_2(b,\varepsilon),b,\varepsilon}, \cdots, U_{a_n(b,\varepsilon),b,\varepsilon}\}.$$

It follows

$$f_b^\varepsilon := \min\{f_{a_1(b,\varepsilon),b}, f_{a_2(b,\varepsilon),b}, \cdots, f_{a_n(b,\varepsilon),b}\} < f + \varepsilon \quad \text{ on } X.$$

[6] For another elementary and more "constructive" proof, see my lecture notes in 2019.

Note that by Observation 2, $f_b^\varepsilon \in \mathcal{A}$. Moreover, by definition, $f_b^\varepsilon(b) = f(b)$. So the sets

$$V_{b,\varepsilon} := \{x \in X \mid f_b^\varepsilon(x) > f(x) - \varepsilon\}$$

is again an open covering of X when we vary b. By compactness, we can find a finite sub-covering

$$\{V_{b_1,\varepsilon}, V_{b_2,\varepsilon}, \cdots, V_{b_m,\varepsilon}\}.$$

It follows

$$f + \varepsilon > f_\varepsilon := \max\{f_{b_1}^\varepsilon, f_{b_2}^\varepsilon, \cdots, f_{b_m}^\varepsilon\} > f - \varepsilon \quad \text{on } X.$$

Using Observation 2 again, we get $f_\varepsilon \in \mathcal{A}$. The proof is completed. □

¶ Stone-Weierstrass Theorem, Version 3.

We can also state the Stone-Weierstrass Theorem in the following form:

Theorem 2.11 (Stone-Weierstrass Theorem for compact Hausdorff space, Version 3)**.** *Let X be compact Hausdorff, and $\mathcal{A} \subset \mathcal{C}(X, \mathbb{R})$ be a subalgebra which separates points. If $\mathcal{A}$ is NOT dense, then there exists a unique $x_0 \in X$ such that*

$$\overline{\mathcal{A}} = \{f \in \mathcal{C}(X, \mathbb{R}) | f(x_0) = 0\}.$$

Proof. Since $\overline{\mathcal{A}} \neq \mathcal{C}(X, \mathbb{R})$, there must exists an x_0 such that $f(x_0) = 0$ for all $f \in \mathcal{A}$. Moreover, such x_0 must be unique since $\mathcal{A}$ separates points. So there exists a unique $x_0 \in X$ with

$$\overline{\mathcal{A}} \subset \{f \in \mathcal{C}(X, \mathbb{R}) \mid f(x_0) = 0\}.$$

Conversely, any $f \in \mathcal{C}(X, \mathbb{R})$ satisfying $f(x_0) = 0$ can be approximated by $f_n \in \mathcal{A}_1$, where $\mathcal{A}_1$ is the unitary subalgebra generated by $\mathcal{A}$ and constant functions. It follows that $f_n - f_n(x_0) \in \mathcal{A}$ and $f_n - f_n(x_0) \to f$ since $f_n(x_0) \to f(x_0) = 0$. The conclusion follows. □

¶ Stone-Weierstrass Theorem for complex-valued functions.

We only considered Stone-Weierstrass theorem for real valued functions above. In general the theorem does not hold for the algebra of complex-valued continuous functions. For example, the algebra of complex polynomials on $\overline{D}$ (the closed unit disc in $\mathbb{C}$) is a unitary complex subalgebra which separate points, but it is not dense in $\mathcal{C}(\overline{D}, \mathbb{C})$, since the function $f(z) = \bar{z}$ cannot be approximated by complex polynomials: If $p_n(z) \to f(z) = \bar{z}$, then we would get

$$0 = \int_0^{2\pi} p_n(e^{it}) e^{it} dt \to \int_0^{2\pi} e^{-it} e^{it} dt = 2\pi,$$

a contradiction. However, if we we assume the subalgebra $\mathcal{A}$ is *self-adjoint*, i.e. closed with respect to conjugation: [7]

$$f \in \mathcal{A} \Longrightarrow \bar{f} \in \mathcal{A},$$

[7] A complex algebra with such a conjugation operation is called a $*$-algebra. We will denote $f^* = \bar{f}$.

then we rescue the theorem:

Theorem 2.12 (Stone-Weierstrass Theorem for complex-valued functions)**.** *Let X be compact Hausdorff, and $\mathcal{A} \subset \mathcal{C}(X, \mathbb{C})$ be a complex subalgebra* [8] *which separates points and vanishes at no point. Moreover, assume $\mathcal{A}$ is self-adjoint, then $\mathcal{A}$ is dense in $\mathcal{C}(X, \mathbb{C})$.*

The proof will be left as an exercise.

¶ A long remark: algebraization of topology.

Back to the case of compact Hausdorff space X. Note that $\mathcal{C}(X, \mathbb{C})$ is a Banach space with respect to the norm

$$\|f\| := d_u(f, 0).$$

Moreover, the product, norm and conjugation are "compatible" in the following sense

$$\|fg\| \le \|f\| \cdot \|g\| \quad \text{ and } \|\bar{f}f\|^2 = \|f\|^2.$$

In general,

Definition 2.13. An C^*-algebra $(\mathcal{A}, +, \cdot, *, \|\cdot\|)$ is a complex algebra $(\mathcal{A}, +, \cdot)$ with an involution $* : \mathcal{A} \to \mathcal{A}$, and a norm $\|\cdot\|$, so that

(1) $(\mathcal{A}, +, \cdot, *)$ is a $*$-algebra, i.e.
   - $x^{**} = x$,
   - $(x+y)^* = x^* + y^*, (xy)^* = y^* x^*$,
   - $(\lambda x)^* = \bar{\lambda} x^*$.

(2) $(\mathcal{A}, +, \cdot, \|\cdot\|)$ is a Banach algebra, i.e. $(\mathcal{A}, +, \|\cdot\|)$ is a Banach space, and $\|xy\| \le \|x\|\|y\|$.

(3) compatibility between the $*$-algebra structure and the Banach structure: $\|x^*x\| = \|x^*\|\|x\|$.

So $\mathcal{C}(X, \mathbb{C})$ is a commutative unitary C^*-algebra. It turns out that not only X determines $\mathcal{C}(X, \mathbb{C})$, but also the commutative C^*-algebra $\mathcal{C}(X, \mathbb{C})$ determines the compact Hausdorff space X:

Theorem 2.14 (Banach-Stone)**.** *Two compact Hausdorff spaces X_1 and X_2 are homeomorphic if and only if $\mathcal{C}(X_1, \mathbb{C})$ and $\mathcal{C}(X_2, \mathbb{C})$ are isomorphic.*

And a remarkable theorem in the theory of operator algebra, proven by Israel Gelfand[9] and Mark Naimark in 1943, claims that any (abstract) unitary commutative

[8] In other words, the scalars a and b in Definition 2.3 are complex numbers now.

[9] Israel M. Gelfand (1913-2009), one of the greatest mathematicians of the 20th century who made significant contributions to many branches of mathematics, including group theory, representation theory and functional analysis, as well as in mathematics education. I will not list his contributions because the list is too long, but only mention that he was awarded the first *Wolf Prize* in 1978, and that he was awarded *Order of Lenin* three times.

C^*-algebra arises in this way and thus gives an explicit construction from $\mathcal{C}(X, \mathbb{C})$ to X:

Theorem 2.15 (Gelfand-Naimark)**.** *Any (abstract) unitary commutative C^*-algebra $\mathcal{A}$ is isomorphic to $\mathcal{C}(X, \mathbb{C})$ for some compact Hausdorff space X.*

Remark 2.16. Here is a rough sketch explaining how to construct such a compact Hausdorff space form a unitary commutative C^*-algebra $\mathcal{A}$: An *character* of $\mathcal{A}$ is defined to be an algebra homomorphism $\phi : \mathcal{A} \to \mathbb{F}$. The space we want to construct is the set of non-zero characters, Σ, on which we endow a topology as follows. Note that $\mathcal{A}$ is a Banach space, and every character is an element in the dual space $\mathcal{A}^*$. Moreover, each character ϕ has (dual) norm ≤ 1 in $\mathcal{A}^*$, i.e. the set of characters Σ is a subset in $\overline{B(\mathcal{A}^*)}$. According to the Banach-Alaoglu theorem in Lecture 10, $\overline{B(\mathcal{A}^*)}$ is a compact Hausdorff space with respect to the weak-* topology. One can show that Σ is weak-* closed, and thus is also compact and Hausdorff.

In other words, one can replace a space (a geometric notion) by an algebra, with no loss. By extending this *duality* between functions and spaces, one may philosophically regard the more complicated noncommutative C^*-algebras as "non-commutative spaces". This leads to a new branch of mathematics: *noncommutative geometry*.