Topology (H) Lecture 12
Lecturer: Zuoqin Wang
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# THE ARZELA-ASCOLI THEOREM

## 1. Five topologies on $\mathcal{C}(X,Y)$

### ¶ Shortcoming of the three topologies.

Let $X$ be a topological space, and $(Y,d)$ a metric space. Last time we have seen three topologies, $\mathscr{T}_{p.c.}, \mathscr{T}_{uniform}, \mathscr{T}_{box}$, on the space of continuous maps,

$$\mathcal{C}(X,Y) = \{f \in \mathcal{M}(X,Y) \mid f \text{ is continuous}\}.$$

We want to study convergence of sequences of functions in $\mathcal{C}(X,Y)$.

*Example* 1.1. Consider the case $X = Y = \mathbb{R}$, then

(1) With respect to the pointwise convergence topology, the sequence of functions $f_n(x) = e^{-nx^2}$ converges in $\mathscr{T}_{p.c.}$ to a bad limit function, the discontinuous function $f_0(x)$ which equals 1 at $x=0$ and equals 0 for all other $x$.
    Underlying reason: The pointwise convergence topology (=the product topology) is too weak for the limit of a convergent sequence of continuous functions to be continuous.

(2) With respect to the uniform convergence topology and the box topology, the sequence of functions $f_n(x) = x^2/n$ would not converge in $\mathscr{T}_{u.c.}$, although it does converge to a nice limit function $f_0(x) \equiv 0$ in the pointwise sense.
    Underlying reason: The uniform topology (and thus the box topology) is too strong for a sequence to converge.

We want to find a reasonable topology on $\mathcal{C}(X,Y)$ so that "bad convergent sequences" are no longer convergent in this topology, while "good convergent sequences" are still convergent. By the analysis above, what we need should be a new topology on $\mathcal{C}(X,Y)$ that is weaker than $\mathscr{T}_{uniform}$, but the limit of a convergent sequence of continuous functions with respect to this new topology is still continuous.

> Observation: Continuity is a "local phenomena" (which is stronger than "pointwise phenomena" and weaker than "global phenomena"). The pointwise convergence is too weak since it is pointwise. The uniform convergence is too strong since it is global. So the correct way is to replace the uniform convergence by its local analogue.

For example, although $f_n(x) = x^2/n \nrightarrow f(x) = 0$ uniformly on $\mathbb{R}$, we do have:

$$\forall [a,b] \subset \mathbb{R}, f_n(x) = x^2/n \longrightarrow f(x) = 0 \text{ \underline{uniformly} on } [a,b].$$

**¶ The compact convergence topology.**

Motivated by the last example above, one may try to find a topology on $\mathcal{M}(X,Y)$ that describes the "convergence on each compact subset". It it not too hard to find one: Let $X$ be a topological space, $(Y,d)$ be a metric space. For any compact set $K \subset X$ and any $\varepsilon > 0$, we denote [Compare: $\omega(f; x_1, \cdots, x_n; \varepsilon)$ in Lecture 4! ]

$$B(f; K, \varepsilon) = \{g \in \mathcal{M}(X,Y) \mid \sup_{x\in K} d(f(x), g(x)) < \varepsilon\}.$$

**Lemma 1.2.** *The family*

$$\mathcal{B}_{c.c.} = \{B(f; K, \varepsilon) \mid f \in \mathcal{M}(X,Y), K \subset X \text{ compact}, \varepsilon > 0\}$$

*is a base of a topology $\mathscr{T}_{c.c}$ on $\mathcal{M}(X,Y)$, which satisfies the following property:*

$$\boxed{f_n \to f \text{ uniformly on each compact set in } X \iff f_n \to f \text{ in } (\mathcal{M}(X,Y), \mathscr{T}_{c.c.}).}$$

*Proof.* The family $\mathcal{B}_{c.c.}$ is a base because for any

$$g \in B(f_1; K_1, \varepsilon_1) \cap B(f_2; K_2, \varepsilon_2),$$

if we take

$$\varepsilon_0 = \min(\varepsilon_1 - \sup_{x\in K_1} d(f_1(x), g(x)),\ \varepsilon_2 - \sup_{x\in K_2} d(f_2(x), g(x))),$$

then we have

$$B(g; K_1 \cup K_2, \varepsilon_0) \subset B(f_1; K_1, \varepsilon_1) \cap B(f_2; K_2, \varepsilon_2),$$

The topology $\mathscr{T}_{c.c.}$ satisfies the demanded property because

$$\begin{aligned} & f_n \to f \text{ uniformly on each compact subset } K \subset X \\ \iff & \forall \varepsilon > 0, \forall \text{ compact } K \subset X, \exists N \text{ s.t. } \sup_{x\in K} d(f_n(x), f(x)) < \varepsilon, \forall n > N \\ \iff & \forall \varepsilon > 0, \forall \text{ compact } K \subset X, \exists N \text{ s.t. } f_n \in B(f; K, \varepsilon), \forall n > N \\ \iff & f_n \to f \text{ in } (\mathcal{M}(X,Y), \mathscr{T}_{c.c.}). \end{aligned}$$

□

**Definition 1.3.** The topology $\mathscr{T}_{c.c.}$ on $\mathcal{M}(X,Y)$ generated by the base $\mathcal{B}_{c.c.}$ is called the *compact convergence topology*.

*Remark* 1.4. By definition, we always have $\mathscr{T}_{product} = \mathscr{T}_{p.c.} \subset \mathscr{T}_{c.c.} \subset \mathscr{T}_{u.c.}$ on $\mathcal{M}(X,Y)$. Moreover, if $X$ is compact, then $\mathscr{T}_{c.c.} = \mathscr{T}_{u.c.}$.

*Remark* 1.5. Let $A \subset X$ be any subset. It is easy to check that the restriction map

$$r_A : \mathcal{M}(X,Y) \to \mathcal{M}(A,Y), \quad f \mapsto f|_A$$

is continuous with respect to all three topologies. Since the restriction of a continuous map to a subset is still continuous, the restriction map

$$r_A : \mathcal{C}(X,Y) \to \mathcal{C}(A,Y), \quad f \mapsto f|_A$$

is continuous with respect to all three topologies: $\mathscr{T}_{p.c.}, \mathscr{T}_{c.c.}, \mathscr{T}_{u.c.}$.

**¶ Compactly generated spaces.**

Back to our original problem. Suppose $f_n \in \mathcal{C}(X,Y)$ and $f_n \to f_0$ w.r.t. $\mathscr{T}_{c.c.}$. By Lemma 1.2, on each compact subset $K \subset X$, we have $f_n \to f_0$ uniformly, and thus $f_0$ is continuous on $K$. Now the problem becomes: under what condition on $X$, "a function $f_0$ is continuous on each compact subset $K \subset X$" implies "$f_0$ is continuous on $X$"? Let $V$ be any open set in $Y$. Then what we want is "$f^{-1}(V)$ is open in $X$", and what we have is "$f^{-1}(V) \cap K$ is open in $K$ for each $K$". Obviously what we need is

($\star$) If $A \cap K$ is open in $K$ for each compact set $K$, then $A$ is open in $X$.

**Definition 1.6.** We say a topological space $X$ is *compactly generated* (or is a *k-space*) if the condition ($\star$) holds.

Obviously in condition ($\star$), we can replace both "open" by "closed".

*Remark* 1.7. With the language of Example 2.11 and Theorem 2.7 in Lecture 5, we get

> $X$ is compactly generated $\iff$ $X$ is the *topological union* of all compact subspaces $K \subset X$, each quipped with the subspace topology.

This explains the name "compactly generated": The topology on $X$ is generated by the topology on all its compact subspaces.

By the analysis above, we conclude

**Proposition 1.8.** *If $X$ is compactly generated, $f_n \in \mathcal{C}(X,Y)$ and $f_n \to f_0$ w.r.t. $\mathscr{T}_{c.c.}$, then $f_0 \in \mathcal{C}(X,Y)$.*

Obviously any compact topological space is compactly generated. In fact, under very mild conditions, $X$ is compactly generated, e.g.

- all first countable spaces (and thus all metric spaces) (exercise)
- all locally compact spaces (will be defined below)
- all CW complexes (important topological spaces in algebraic topology. [1] )

are compactly generated. Here we give an example that is NOT compactly generated:

*Example* 1.9. Consider the space $\mathcal{M}(\mathbb{R},\mathbb{R})$, endowed with the pointwise convergence topology. Consider the subset $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n := \{f \mid \exists \text{ subset } S \subset \mathbb{R} \text{ with } |S| \le n \text{ s.t. } f(x) = 0 \text{ on } S \text{ and } f(x) = n \text{ on } \mathbb{R} \setminus S.\}$$

Then one has [We use 0 to represent the constant function $f(x) \equiv 0$.]

- $\overline{A} = A \cup \{0\}$. In particular, since $0 \notin A$, we conclude that $A$ is not closed.
  Obviously $0 \in A' \setminus A$. Conversely suppose $f \in A' \setminus A$. If $\mathrm{Im}(f) \not\subset \mathbb{N}_{>0} \cup \{0\}$ or $\mathrm{Im}(f)$ contains at least two different values in $\mathbb{N}_{>0}$, it is easy to construct a neighborhood $U$ of $f$ so that $U \subset A^c$. If $\mathrm{Im}(f) = \{0, n\}$ for some $n \in \mathbb{N}_{>0}$, then

[1] c.f. A. Hatcher, *Algebraic Topology*, Appendix.

either (in the case $|f^{-1}(0)| \le n$) we have $f \in A$, or (in the case $|f^{-1}(0)| > n$ ) one can find a neighborhood $U$ of $f$ so that $U \subset A^c$. For the case $\text{Im}(f) = \{n\}$ for some $n \in \mathbb{N}_{>0}$ we have $f \in A$. So if $f \in A' \setminus A$ we must have $f \equiv 0$.

- For any compact set $K \subset \mathcal{M}(\mathbb{R}, \mathbb{R})$, $A \cap K$ is compact.

  To see this, we first notice that (since $\mathcal{M}(\mathbb{R}, \mathbb{R})$ is Hausdorff) $K$ must be closed and thus $\overline{A} \cap K$ is compact. So the conclusion holds if $0 \notin K$. Now assume $0 \in K$. Since any open covering of $\overline{A} \cap K$ has a finite covering, there exists $N$ such that $\overline{A} \cap K \subset \{0\} \cup \bigcup_{k=1}^N A_k$. Now take any open covering $\{U_\alpha\}$ of $A \cap K$, by adding one open set $U_0 = \omega(0; x_1, \cdots, x_{N+1})$, we get an open covering of $\overline{A} \cap K$ which has a finite subcovering $\{U_0, U_1, \cdots, U_m\}$. Since $\omega(0; x_1, \cdots, x_{N+1}) \cap (A \cap K) \subset \omega(0; x_1, \cdots, x_{N+1}) \cap A = \emptyset$, $\{U_1, \cdots, U_m\}$ is a finite subcovering of $\{U_\alpha\}$ that covers $A \cap K$. So $A \cap K$ is compact.

So $(\mathcal{M}(\mathbb{R}, \mathbb{R}), \mathcal{T}_{p.c.})$ is not compactly generated.

**¶ Locally compact Hausdorff spaces.**

Now we introduce an important class of topological spaces:

**Definition 1.10.** We say a topological space $X$ is *locally compact* if every point $x \in X$ has a compact neighbourhood, i.e., there exists an open set $U$ and a compact set $K$ such that $x \in U \subset K$.

**Proposition 1.11.** *Any locally compact space is compactly generated.*

*Proof.* Suppose $X$ is locally compact. Suppose $A \subset X$ and $A \cap K$ is open in $K$ for any compact subset $K \subset X$. To prove $A$ is open, for any $x \in A$ we take an open set $U$ and a compact set $K$ such that $x \in U \subset K$. Then $A \cap K$ is open in $K$. It follows that $A \cap U$ is open in $U$ and thus $A \cap U$ is open in $X$. So $A$ is open. □

In almost all applications, locally compact spaces are also Hausdorff. We will call a locally compact Hausdorff space a *LCH space*. They play an important role in analysis. For example, Real analysis on $\mathbb{R}^n$ (measure theory and integration) can be extended to LCHs.[2] (One can prove that the space $\mathbb{Q}_p$, i.e. the completion of $\mathbb{Q}$ under the $p$-adic metric, is a LCH. As a result, analysis on LCH is very useful in $p$-adic analysis.

Here are some examples of LCH and non-LCH:

*Example* 1.12.

- Any compact Hausdorff space is LCH.
- $\mathbb{R}^n$ is LCH. More generally, any locally Euclidian[3] Hausdorff space is LCH.
- Neither $\mathbb{Q} \subset \mathbb{R}$ nor $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$ is locally compact. [Why?]

[2]For a reference, see G. Folland, *Real analysis*, Chapter 7, or T. Tao, *An Epsilon of Room I: Real Analysis*, Section 1.10.

[3]We say a topological space is *locally Euclidian* if for any $x \in X$, there exists a neighborhood $U$ of $x$ which is homeomorphic to an open ball in Euclidian space.

We will need the following proposition whose proof is left as an exercise:

**Proposition 1.13.** *Let $X$ be a LCH, $K$ be a compact set in $X$, and $U$ be an open set in $X$ such that $K \subset U$. Then there exists an open set $V$ such that $\overline{V}$ is compact, and*

$$K \subset V \subset \overline{V} \subset U.$$

**¶ The compact open topology.**

The compact convergence topology is defined for maps from a topological space to a metric space. It is easy to extend the definition of compact convergence topology to a topology on $\mathcal{M}(X,Y)$, where both $X$ and $Y$ are topological spaces:

**Definition 1.14.** Let $X, Y$ be topological spaces. For any compact $K \subset X$ and open $V \subset Y$, we denote

$$S(K,V) = \{f \in \mathcal{M}(X,Y) \mid f(K) \subset V\}.$$

The topology $\mathscr{T}_{c.o.}$ on $\mathcal{M}(X,Y)$ generated by the sub-base

$$\mathcal{S}_{c.o.} = \{S(K,V) \mid K \subset X \text{ compact }, V \subset Y \text{ open }\}$$

is called the *compact-open topology.*

We are only interested in $\mathscr{T}_{c.o.}$ on $\mathcal{C}(X,Y)$, since it is most useful in this subspace.

*Example* 1.15. If we take $X$ to be a single point set $\{*\}$, then the space $(\mathcal{C}(\{*\},Y), \mathscr{T}_{c.o.})$ is homeomorphic to the space $Y$ itself.

*Remark* 1.16. One can prove (exercise) that if $Y$ is a metric space, then $\mathscr{T}_{c.o.} = \mathscr{T}_{c.c.}$ on $\mathcal{C}(X,Y)$. In particular, the topology $\mathscr{T}_{c.c.}$ on $\mathcal{C}(X,Y)$ is independent of the choice of topologically equivalent metrics on $Y$. (So if $X$ is compact, then $\mathscr{T}_{u.c.}$ on $\mathcal{C}(X,Y)$ is independent of the choice of topologically equivalent metrics on $Y$.)

It turns out that with respect to the compact-open topology, the composition is continuous as long as the "middle variable space" is locally compact:

**Proposition 1.17.** *Suppose $X, Y$ and $Z$ are topological spaces, where $Y$ is locally compact Hausdorff. Then the composition map*

$$\circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z), \quad (f,g) \mapsto g \circ f$$

*is continuous (with respect to the compact-open topologies on each space).*

The proof is based on Proposition 1.13 and is also left as an exercise.

**Corollary 1.18.** *Let $X$ be a locally compact Hausdorff space, and $Y$ be any topological space. Then the evaluation map*

$$e : X \times \mathcal{C}(X,Y) \to Y, \quad (x,f) \mapsto e(x,f) = f(x) \in Y$$

*is continuous when we endow $\mathcal{C}(X,Y)$ with the compact-open topology.*

*Proof.* Identifying $X$ with $\mathcal{C}(\{*\},X)$ and $Y$ with $\mathcal{C}(\{*\},Y)$. Then the evaluation map is just the composition map. □

## 2. Arzela-Ascoli theorem

**¶ The classical Arzela-Ascoli theorem.**

Given a sequence, or more generally, a family of continuous maps, one of the central problem in analysis is: Can one find a subsequence that converges (uniformly) to another continuous function?

For example, in analysis, to prove the existence of a solution to a PDE or a variational problem, one can first try to construct a sequence of functions which solve the problem "approximately". If one can show the sequence of "approximate solutions" has a subsequence that converges to a nice function, then usually with some extra work, one can show that the limit is in fact a true solution. Such a method is known as a "compactness argument". One of the most useful tool to carry out such a compactness argument for functions is the Arzela-Ascoli theorem. As we mentioned in Lecture 1, one motivation for the Arzela-Ascoli theorem is to rescue Dirichlet's principle, i.e. trying to prove the existence of a solution to the Laplace equation $\Delta u(x, y, z) = 0$ with prescribed boundary conditions.

The original version of Arzela-Ascoli theorem that you may have seen in your analysis course is

**Theorem 2.1** (Arzela-Ascoli, classical version)**.** *If a sequence $\{f_n\} \in \mathcal{C}([0,1],\mathbb{R})$ is uniformly bounded and equicontinuous, then it has a convergence subsequence.*

**¶ Equicontinuity.**

Recall that a family of functions $\mathcal{F} \subset \mathcal{C}([0,1],\mathbb{R})$ is

- *uniformly bounded* if there exists $M > 0$ such that $\forall x \in [0,1]$ and $\forall f \in \mathcal{F}$,
$$|f(x)| \leq M.$$
- *equicontinuous* if for any $x_0 \in [0,1]$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in [0,1]$ with $|x - x_0| < \delta$ and for all $f \in \mathcal{F}$, we have
$$|f(x) - f(x_0)| \leq \varepsilon.$$

It is not hard to see that the two conditions are necessary:

(1) The sequence $f_n(x) = n$ is equicontinuous but has no convergent subsequence because it is not uniformly bounded (although each function in the sequence is a bounded function).
(2) The sequence $f_n(x) = x^n$ is uniformly bounded on $[0,1]$ but has no convergent subsequence (in $\mathcal{C}([0,1],\mathbb{R})$ because it is not equicontinuous at $x = 1$ (although each function in the sequence is continuous at $x = 1$).

The conception of equicontinuity can be easily generalized to maps from an arbitrary topological space $X$ to a metric space $Y$:

**Definition 2.2.** Let $(Y,d)$ be a metric space, and $X$ be a topological space. Let $\mathcal{F} \subset \mathcal{C}(X,Y)$ be a subset. We say $\mathcal{F}$ is *equicontinuous* at $x_0 \in X$ if for any $\varepsilon > 0$, there exists an open neighbourhood $U$ of $x_0$ such that

$$d(f(x), f(x_0)) < \varepsilon, \quad \forall x \in U, \forall f \in \mathcal{F}.$$

We say $\mathcal{F}$ is *equicontinuous* if it is equicontinuous at any point $x \in X$.

Note that equicontinuity is a metric property: it depends on the metric on $Y$. It turns out that equicontinuity is a generalization of totally boundedness in $(\mathcal{C}(X,Y), d_u)$:

**Proposition 2.3.** *Let $(Y,d)$ be a metric space, and $\mathcal{F}$ be a totally bounded subset in $\mathcal{C}(X,Y)$ (with respect to $d_u$). Then $\mathcal{F}$ is equicontinuous.*

*Proof.* For any $x_0 \in X$ and $\varepsilon > 0$, we need to find an open neighborhood $U$ of $x_0$ s.t.

$$d(f(x), f(x_0)) < \varepsilon, \quad \forall x \in U, \forall f \in \mathcal{F}.$$

Since $\mathcal{F}$ is totally bounded, there exists a finite $\frac{\varepsilon}{4}$-net $\{f_1, \cdots, f_n\}$ of $\mathcal{F}$ in $(\mathcal{C}(X,Y), d_u)$. Since each $f_k$ is continuous, the set

$$U = \bigcap_{k=1}^{n} f_k^{-1}\left(B(f_k(x_0), \frac{\varepsilon}{3})\right)$$

is an open neighborhood of $x_0$. Now for any $f \in \mathcal{F}$, by our choice there exists $k$ such that $d_u(f, f_k) < \varepsilon/4$. It follows that for any $x \in U$,

$$d(f(x), f(x_0)) \le d(f(x), f_k(x)) + d(f_k(x), f_k(x_0)) + d(f_k(x_0), f(x_0)) < \varepsilon.$$

This completes the proof. □

We know that $\mathcal{C}(X,Y)$ is not closed with respect to $\mathscr{T}_{p.c.}$. However, for an equicontinuous family $\mathcal{F} \subset \mathcal{C}(X,Y)$, we have

**Proposition 2.4.** *Let $\mathcal{F} \subset \mathcal{C}(X,Y)$ be equicontinuous. Let $\mathcal{K}$ be the closure of $\mathcal{F}$ with respect to $\mathscr{T}_{p.c.}$. Then $\mathcal{K}$ is equicontinuous (and in particular $\mathcal{K} \subset \mathcal{C}(X,Y)$).*

*Proof.* For any $x_0 \in X$ and $\varepsilon > 0$, we need an open neighborhood $U$ of $x_0$ s.t.

$$(*) \qquad d(g(x), g(x_0)) < \varepsilon, \quad \forall x \in U, \ \forall g \in \mathcal{K}.$$

By the equicontinuity of $\mathcal{F}$, we can find an open neighbourhood $U$ of $x_0$ s.t.

$$d(f(x), f(x_0)) < \frac{\varepsilon}{3}, \quad \forall x \in U, \ \forall f \in \mathcal{F}.$$

To prove (*) for this $U$, we fix any $g \in \mathcal{K}$, $x \in U$ and denote

$$\begin{aligned} V &= \left\{h \in Y^X \,\middle|\, d(h(x), g(x)) < \frac{\varepsilon}{3}, \ d(h(x_0), g(x_0)) < \frac{\varepsilon}{3}\right\} \\ &= \pi_x^{-1}\left((g(x) - \frac{\varepsilon}{3}, g(x) + \frac{\varepsilon}{3})\right) \bigcap \pi_{x_0}^{-1}\left((g(x_0) - \frac{\varepsilon}{3}, g(x_0) + \frac{\varepsilon}{3})\right). \end{aligned}$$

Then $V$ is an open neighbourhood of $g$ in $Y^X$. Since $g \in \mathcal{K}$ and $\mathcal{K}$ is the closure of $\mathcal{F}$ in $Y^X$, we have $V \cap \mathcal{F} \neq \emptyset$. Take any $f \in V \cap \mathcal{F}$, we get

$$d(g(x), g(x_0)) \leqslant d(g(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), g(x_0)) < \varepsilon.$$

This proves (*) and thus the equicontinuity of $\mathcal{K}$. □

**¶ Arzela-Ascoli theorem, the general version.**

We want $\mathcal{F}$ to satisfy the property that any sequence in $\mathcal{F}$ has a (uniformly) convergent subsequence which converges to a function in $\mathcal{C}(X, Y)$ (but the limit may be outside $\mathcal{F}$). In other words, we want the closure $\overline{\mathcal{F}}$ to be compact or is contained in a compact set in $\mathcal{C}(X, Y)$ with respect to $\mathscr{T}_{c.c.}$ (or $\mathscr{T}_{u.c.}$ if you want uniform convergence).

**Definition 2.5.** A subset $A$ in a topological space $X$ is called *precompact* (or *relatively compact*) if $\overline{A}$ is compact.

For simplicity we also introduce the following definition:

**Definition 2.6.** We say a family $\mathcal{F} \subset \mathcal{C}(X, Y)$ is

(1) *pointwise bounded* if $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ is bounded in $Y$ for each $a \in X$,
(2) *pointwise precompact* if $\mathcal{F}_a$ is precompact in $Y$ for each $a \in X$.

Today we are going to prove the following general form[4] of Arzela-Ascoli theorem:

**Theorem 2.7** (Arzela-Ascoli theorem, the general version)**.** *Let $X$ be a topological space and $(Y, d)$ a metric space. Let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$ which is endowed with the compact-convergence topology $\mathscr{T}_{c.c.}$.*

*(1) Suppose $\mathcal{F}$ is equicontinuous and pointwise precompact. Then the closure of $\mathcal{F}$ is compact in $(\mathcal{C}(X, Y), \mathscr{T}_{c.c.})$.*
*(2) If $X$ is locally compact and Hausdorff, then the converse holds.*

**¶ Arzela-Ascoli theorem, the general version: the proof.**

Now let's prove the main theorem. Although the theorem is about the compact convergence topology $\mathscr{T}_{c.c.}$, we do use $\mathscr{T}_{p.c.}$ and $\mathscr{T}_{u.c}$ in the proof as well.

**Idea:** We denote $\overline{\mathcal{F}}^{c.c.}$ = the closure of $\mathcal{F}$ with respect to $\mathscr{T}_{c.c.}$. We want to prove that $\overline{\mathcal{F}}^{c.c.}$ is compact in $(\mathcal{C}(X, Y), \mathscr{T}_{c.c.})$. The idea is to start with another space in which it is easier to deduce compactness. [The auxiliary space should has less open sets such that it is easier for a set to be compact. The best candidate is the pointwise convergence topology (= the product topology), since we have the wonderful Tychonoff theorem.] So we denote $\mathcal{K} := \overline{\mathcal{F}}^{p.c.}$ = the closure of $\mathcal{F}$ with respect to $\mathscr{T}_{p.c.}$, and so in Step 1 we will show that $\mathcal{K}$ is compact with respect to $\mathscr{T}_{p.c.}$. Then the strategy to prove (1) is to show that the two topologies, $\mathscr{T}_{p.c.}$ and $\mathscr{T}_{c.c.}$ coincides on $\mathcal{K}$. Since $\mathscr{T}_{c.c.}$ is finer

[4]There exists even more general form of Arzela-Ascoli theorem which characterize the compactness of family of maps into a uniform space (which is a generalization of metric space).

than $\mathscr{T}_{p.c.}$ in general, we have to show the reverse inclusion relation when restricted to $\mathcal{K}$. For this purpose, we need to show that for two functions in $\mathcal{K}$ to be $\varepsilon$-close on a compact set $K$ (and thus "close" in $\mathscr{T}_{c.c.}$), it is enough that they are $\varepsilon$-close on a finite set (i.e. "close" in $\mathscr{T}_{p.c.}$). To prove this, we need the equicontinuity of $\mathcal{K}$ proven in Proposition 2.4 and use compactness of $K$ to get such a finite set (in Step 2).

Notation: Although as a set we have $\mathcal{M}(X,Y) = Y^X$, we will distinguish these two notions in this proof: we write $\mathcal{M}(X,Y)$ or $\mathcal{C}(X,Y)$ when we use the compact convergence topology, and when we write $Y^X$ we will use the pointwise convergence topology. So in the following proof, $\mathcal{C}(X,Y)$ is NOT a topological subspace of $Y^X$, although it is a subset.

*Proof of Theorem 2.7.*

(1) We denote $\mathcal{K}=$ the closure of $\mathcal{F}$ in $Y^X$.

**Step1:** $\mathcal{K}$ is compact in $Y^X$.

Let $K_a = \overline{\mathcal{F}_a}$ in $Y$. Then $K_a$ is compact by assumption, and is closed since $Y$ is a metric space and thus is Hausdorff. So

$$\prod_{a\in X} K_a = \bigcap_{a\in X} \pi_a^{-1}(K_a)$$

is compact (by Tychonoff) in $Y^X$, and is closed since $Y^X$ is Hausdorff. Since

$$\mathcal{F} \subset \prod_{a\in X} \mathcal{F}_a \subset \prod_{a\in X} K_a,$$

its closure $\mathcal{K}$, as a closed subset in the compact set $\prod_{a\in X} K_a$, is compact in $Y^X$.

**Step2:** The two topologies $\mathscr{T}_{p.c.}$ and $\mathscr{T}_{c.c.}$ coincide on $\mathcal{K}$.

[Once this is done, then together with Step 1, we conclude that $\mathcal{K}$ is the closure of $\mathcal{F}$ in $(\mathcal{C}(X,Y), \mathscr{T}_{c.c.})$ and is compact. This proves (1). ]

Since we always have $\mathscr{T}_{p.c.} \subset \mathscr{T}_{c.c.}$, it is enough to prove the reverse on $\mathcal{K}$. In other words, we need to show: for any $g \in \mathcal{K}$ and for any $B(g;K,\varepsilon) \subset \mathcal{C}(X,Y)$, where $K \subset X$ is compact, there exists an open set $U \subset Y^X$ with $g \in U$ s.t.

$$U \cap \mathcal{K} \subset B(g;K,\varepsilon) \cap \mathcal{K}. \tag{**}$$

By Proposition 2.4 $\mathcal{K}$ is equicontinuous, together with the fact that $K$ is compact, we can find finitely many points $x_1, \cdots, x_n$ and open sets $V_1, \cdots, V_n$ in $X$ covering $K$, such that

$$d(\tilde{g}(x), \tilde{g}(x_i)) < \frac{\varepsilon}{3}, \quad \forall \tilde{g} \in \mathcal{K},\ \forall x \in V_i.$$

So we take $U$ to be the set

$$U = \omega(g; x_1, \cdots, x_n, \varepsilon) = \left\{ h \in Y^X | d(h(x_i), g(x_i)) < \frac{\varepsilon}{3}, 1 \leqslant i \leqslant n \right\}.$$

It is easy to check (**) holds for this $U$:

Suppose $h \in U \cap \mathcal{K}$. For any $x \in K$, there exists $i$ s.t. $x \in V_i$. So

$$\begin{aligned} d(h(x), g(x)) &\leqslant d(h(x), h(x_i)) + d(h(x_i), g(x_i)) + d(g(x_i), g(x)) \\ &< \varepsilon, \quad \forall x \in K. \end{aligned}$$

In other words, $h \in B(g; K, \varepsilon)$.

This completes Step 2 and thus proves (1).

(2) Now suppose $X$ is LCH, and suppose the closure $\mathcal{K}$ of $\mathcal{F}$ in $\mathcal{C}(X, Y)$ is compact. We will show $\mathcal{K}$ is equicontinuous and pointwise compact, which would imply that $\mathcal{F}$ is equicontinuous and pointwise pre-compact (since each $\overline{\mathcal{F}_a}$ is a closed subset in $\mathcal{K}_a$).

The compactness of $\mathcal{K}_a$ follows from Corollary 1.18: $\mathcal{K}_a$ is the image of the compact set $\mathcal{K}$ under the continuous map

$$\mathcal{C}(X, Y) \xrightarrow{j_a} X \times \mathcal{C}(X, Y) \xrightarrow{e} Y$$

and thus is compact, where $j_a$ is the "embedding map" $j_a(f) := (a, f)$.

To show the equicontinuity of $\mathcal{K}$ at an arbitrary $x \in X$, we take a compact neighborhood $A$ of $x$. Then it is enough to show

$$\mathcal{K}_A := \{r_A(f) \mid f \in \mathcal{K}\}$$

is equicontinuous at $x$, where $r_A : \mathcal{C}(X, Y) \to \mathcal{C}(A, Y)$ is the restriction map. Since $r_A$ is continuous, $\mathcal{K}_A = r(\mathcal{K})$ is compact in $\mathcal{C}(A, Y)$. Since $A$ is compact, the compact convergence topology coincides with the uniform convergence topology on $\mathcal{C}(A, Y)$. So compactness of $\mathcal{K}_A$ in $\mathcal{C}(A, Y)$ implies that $\mathcal{K}_A$ is totally bounded with respect to $d_u$. By Proposition 2.3, $\mathcal{K}_A$ is equicontinuous. This completes the proof.

□

*Remark* 2.8. In proving (2) we only used a weaker condition: $\mathcal{F}$ is contained in a compact set $\mathcal{K}$ in $\mathcal{C}(X, Y)$.

**¶ Some special cases and applications.**

Note that the conclusion is quite weak in this very general version, because in general the topology $\mathscr{T}_{c.c.}$ need not be metrizable, and compactness does not imply sequential compactness. Thus for a sequence which is equicontinuous and pointwise precompact, we can't even conclude the existence of a convergent subsequence. However, there are many interesting/important special cases where we are able to conclude the existence of a convergent subsequence:

(a) We know that if $X$ is compact and $Y$ is a metric space, then $\mathscr{T}_{c.c.} = \mathscr{T}_{u.c}$ on $\mathcal{C}(X, Y)$. Since $\mathscr{T}_{u.c.}$ is a metric topology, compactness does imply sequential compactness. So in particular we get

**Theorem 2.9** (Arzela-Ascoli for maps on compact spaces). *Let $X$ be compact and $(Y, d)$ a metric space. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be a subset which is equicontinuous and pointwise precompact. Then any sequence in $\mathcal{F}$ has a subsequence that converges* uniformly *(to a continuous map) on $X$.*

Since in $\mathbb{R}^n$, a set is pre-compact if and only if it is bounded, we get

**Corollary 2.10** (Arzela-Ascoli for functions on compact spaces). *Let $X$ be compact. Let $\mathcal{F} \subset \mathcal{C}(X, \mathbb{R}^n)$ be a subset which is equicontinuous and pointwise bounded. Then any sequence in $\mathcal{F}$ has a subsequence that converges uniformly (to a bounded continuous function) on $X$.*

(b) For the case of locally compact space, every point has a compact neighborhood. Obviously if a family $\mathcal{F}$ is equicontinuous/pointwise precompact, then its restriction to such a neighborhood is also equicontinuous/pointwise precompact. So if $X$ is locally compact, then for any sequence $\{f_n\}$ which is equicontinuous and pointwise compact, and for any point $x$, there is a compact neighborhood of $x$ on which $\{f_n\}$ has a convergent subsequence. Unfortunately this is not strong enough to claim that the sequence $\{f_n\}$ has a convergent subsequence in $\mathscr{T}_{c.c}$, because there may be "too much compact subsets" in $X$. However, if we assume $X$ is *$\sigma$-compact*, i.e. $X$ is a countably union of compact subsets, then we may apply the standard diagonalization trick to extract a subsequence which converges (uniformly!) on each compact subset:

**Theorem 2.11** (Arzela-Ascoli for maps on locally compact and $\sigma$-compact spaces). *Let $X$ be locally compact and $\sigma$-compact, and $(Y, d)$ be a metric space. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be a subset which is equicontinuous and pointwise precompact. Then any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact sets of $X$ to a limit function $f \in \mathcal{C}(X, Y)$.*

*Remark* 2.12. The Arzela-Ascoli theorem is widely used in analysis. Here are some standard applications that you can learn from other courses:

- Functional analysis: Frechet-Kolmogorov-Riesz compactness theorem.
- PDE: Sobolev embedding etc.
- ODE: Peano existence theorem.
- Complex analysis: Montel's theorem
- Harmonic analysis/Lie Theory: Peter-Weyl theorem.

Finally, we point out that by using some standard facts in convex geometry, one can prove the following useful compactness theorem due to Wilhelm Blaschke (1885-1962) (he is Chern's teacher), in convex geometry. For more details, c.f. my 2020 notes.

**Theorem 2.13** (Blaschke selection theorem). *For any $R > 0$, the set of all nonempty compact convex subsets contained in $B(0, R)$ is compact (with respect to $\mathscr{T}_{d_H}$).*