Topology (H) Lecture 13
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THE AXIOMS OF COUNTABILITY

1. Axioms of countability

As we have seen, compactness can be regard as a "continuous version" of finiteness. We can think of compact spaces as those spaces that are built-up using only finitely many "building blocks" in topology, namely, open sets. Finiteness is important because it allows us to construct things by hand, and as a result we combine local results to global results for compact spaces.

Now we turn to countability features in topology. In topology, an axiom of countability is a topological property that asserts the existence of a countable set with certain properties. There are several different topological properties describing countability.

¶ First countable spaces.

Let's recall (from Lecture 7)

Definition 1.1. A topological space $(X, \mathscr{T})$ is called *first countable*, or an *(A1)-space*, if it has a countable neighbourhood basis, i.e.

$$\text{(A1)} \qquad \begin{array}{l} \text{For any } x \in X, \text{ there exists a countable family of open} \\ \text{neighborhoods of } x, \{U_1^x, U_2^x, U_3^x, \cdots\}, \text{ such that for any} \\ \text{open neighborhood } U \text{ of } x, \text{ there exists } n \text{ s.t. } U_n^x \subset U. \end{array}$$

Remark 1.2. If $(X, \mathscr{T})$ is first countable, then for each point one can choose a countable neighborhood base $\{U_n^x\}$ satisfying

$$U_1^x \supset U_2^x \supset U_3^x \supset \cdots,$$

since if one has a countable neighbourhood base $V_1^x, V_2^x, \cdots$ at x, then one can take

$$U_1^x = V_1^x, \ \ U_2^x = V_1^x \cap V_2^x, \ \ U_3^x = V_1^x \cap V_2^x \cap V_3^x, \ \cdots.$$

We have seen

Proposition 1.3. *Suppose $(X, \mathscr{T})$ is first countable.*

(1) A subset $F \subset X$ is closed if and only if it contains all its sequential limits, i.e. for any sequence $\{x_n\} \subset F$ with $x_n \to x \in X$, we have $x \in F$. [Lecture 7]
   - *$\rightsquigarrow$ a map $f : X \to Y$ is continuous $\Longleftrightarrow$ it is sequentially continuous.*

(2) If $(X, \mathscr{T})$ is also Hausdorff, then a subset A in X is limit point compact if and only if it is sequentially compact. [Lecture 9]

Here are some examples of (A1)-spaces and non-(A1)-spaces.

Example 1.4.

(1) Any metric space is first countable since we can take $U_n^x = B(x, \frac{1}{n})$.
(2) The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is first countable with $U_n^x = [x, x + \frac{1}{n})$.
(3) The space $(\mathbb{R}, \mathscr{T}_{cocountable})$ is NOT first countable: For any sequence of open neighborhoods $\{U_x^n\}$ of x, one can always construct a new open set by dropping of one more point from $\cap_n U_x^n$, which cannot contain any U_x^n.
(4) The space $(\mathcal{M}([0,1], \mathbb{R}) = \mathbb{R}^{[0,1]}, \mathscr{T}_{product})$ is NOT first countable, since we have seen (Lecture 7) that there exists a non-closed subset
$$A = \{f : [0,1] \to \mathbb{R} \mid f(x) \neq 0 \text{ for only countably many } x\}$$
which contains all its sequential limit points.

¶ Second countable spaces.

For the Euclidean space $\mathbb{R}^n$, we have seen in Lecture 5 that not only it has a countable neighborhood basis at each point x, but it has a basis which contains only countably many open sets,
$$\mathcal{B} = \{B(x,r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0}\}.$$
This is a stronger countability property, which deserve a name: [PSet3-1-4]

Definition 1.5. A topological space $(X, \mathscr{T})$ is called *second countable*, or an *(A2)-space*, if it has a countable basis, i.e.

$$\text{(A2)} \qquad \begin{array}{l} \text{there exists a countable family of open sets } \{U_1, U_2, U_3, \cdots\} \\ \text{which form a base of the topology } \mathscr{T}. \end{array}$$

Obviously any second countable space is a first countable space. But the converse is not true, for example, $(\mathbb{R}, \mathscr{T}_{discrete})$ is first countable as it is a metric space, but it is not second countable.

There is a big class of metric spaces which is second countable:

Proposition 1.6. *Any totally bounded metric space is second countable.*

Proof. Suppose (X,d) is totally bounded. By definition, for any n, one has a finite $\frac{1}{n}$-net, i.e. there exists finitely many points $x_{n,1}, x_{n,2}, \cdots, x_{n,k(n)} \in X$ such that
$$X = \bigcup_{i=1}^{k(n)} B(x_i, \frac{1}{n}).$$
We claim that the countable collection
$$\mathcal{B} := \{B(x_{n,i}, 1/n) : n \in \mathbb{N}, 1 \leqslant i \leqslant k(n)\}$$

is a basis of the metric topology $\mathscr{T}$. To see this, we take any open set U and any point $x \in U$. Then $\exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset U$. Now we choose $n \in \mathbb{N}$ and $1 \leq i \leq k(n)$ s.t.

$$1/n < /\varepsilon/2 \quad \text{and} \quad d(x, x_{n,i}) < 1/n.$$

It follows

$$B(x_{n,i}, \frac{1}{n}) \subset B(x, \frac{2}{n}) \subset B(x, \varepsilon) \subset U,$$

i.e. the countable family $\mathcal{B}$ is a basis. □

Since any compact metric space is totally bounded, we get as a consequence,

Corollary 1.7. *Any compact metric space is second countable.*

Example 1.8. Consider the space

$$X = [0,1]^{\mathbb{N}} = \{(x_1, x_2, \cdots) \mid x_i \in [0,1]\}.$$

- $(X, \mathscr{T}_{product})$ is second countable.
  Reason: Endow X with the metric
  $$d((x_n),(y_n)) := \sum_{n\geq 1} 2^{-n}|x_n - y_n|.$$
  So on X we have a metric topology. Just as PSet6-1-2, The metric topology $\mathscr{T}_{metric}$ and the product topology $\mathscr{T}_{product}$ on X coincide. As a consequence, $(X, \mathscr{T}_{product})$ is a compact metric space, and thus is second countable. (We will also call it Hilbert cube, since it is homeomorphic to the Hilbert cube in Lec 2)
- $(X, \mathscr{T}_{box})$ is not first countable (and thus not second countable).
  Reason: Fix any point $x = (x_i)$ in X. Suppose on the contrary that
  $$\{U_n(x) = \prod_i U_i^{(n)}(x_i) \mid n \in \mathbb{N}\}$$
  is a countable neighborhood base of $(X, \mathscr{T}_{box})$ at $x = (x_i)$. Note that each $U_i^{(n)}(x_i)$ is an open neighborhood of x_i in $[0,1]$. Let $\widetilde{U}_i^{(i)}(x_i) \subsetneqq U_i^{(i)}(x_i)$ be a strictly smaller open neighborhood of x_i in $[0,1]$. Then the set
  $$U := \prod_i \widetilde{U}_i^{(i)}(x_i)$$
  is an open neighborhood of the point (x_n) in the box topology, but none of the $U_n(x)$'s is contained in U, a contradiction.

¶ Separable spaces.

If you think very hard about the countable basis for the Euclidean space $\mathbb{R}^n$ that we constructed, namely

$$\mathcal{B} = \{B(x,r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0}\},$$

you will find that a crucial reason is that $\mathbb{R}^n$ admits a countable dense subset $\mathbb{Q}^n$. It turns out that this is the common feather for any second countable space:

Proposition 1.9. *Any second countable topological space has a countable dense subset.*

Proof. Let $\{U_n|n \in \mathbb{N}\}$ be a countable basis of $(X, \mathscr{T})$. For each n, we choose a point $x_n \in U_n$ and let $A = \{x_n|n \in \mathbb{N}\}$. Then A is a countable subset in X. We claim that $\overline{A} = X$. In fact, for any $x \in X$ and any open neighborhood U of x, there exists n s.t. $x \in U_n \subset U$. In particular, $U \cap A \neq \emptyset$. So we get $\overline{A} = X$. □

Remark 1.10. What we really proved is: In any topological space, there exists a dense subset whose cardinality is no more than the cardinality of a basis. [Note: We used the *axiom of countable choice* in the proof above. For this general version, we need the axiom of choice.]

Definition 1.11. A topological space $(X, \mathscr{T})$ is *separable* if it contains a countable dense subset.

So we can rewrite the proposition we just proved as

$$\boxed{\text{Any second countable topological space is separable.}}$$

The converse is NOT true. For example,

Example 1.12. $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is separable, but not second countable.

- To show $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is separable, it is enough to show $\overline{\mathbb{Q}} = \mathbb{R}$ with respect to the Sorgenfrey topology, which follows from the fact that for any $x \in \mathbb{R}$ and any interval $[x, x+\varepsilon)$, one can find a rational number $r \in [x, x+\varepsilon)$.
- To see $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is not second countable, we let $\mathcal{B}$ be any basis of $\mathscr{T}_{sorgenfrey}$. Then $\forall x \in \mathbb{R}$, there exists an open set $B_x \in \mathcal{B}$ s.t.
$$x \in B_x \subset [x, x+1),$$
which implies $x = \inf B_x$. As a consequence, for any $x \neq y$, we have $B_x \neq B_y$. So $\mathcal{B}$ is an uncountable family.

However, we have

Proposition 1.13. *A metric space is second countable if and only if it is separable.*

Proof. Let (X, d) be a separable metric space, and $A = \{x_n|n \in \mathbb{N}\}$ be a countable dense subset. Then $\mathcal{B} = \{B(x_n, 1/m)|n, m \in \mathbb{N}\}$ is a countable base for the metric topology. □

Remark 1.14. Separability is a very useful conception in functional analysis. It is used to prove certain compactness results. Another well-known result is

A Hilbert space $\mathcal{H}$ is separable $\Longleftrightarrow$ it has a countable orthonormal basis.

From this fact it is easy to construct non-separable Hilbert spaces. For example, let
$$\widetilde{l^2(\mathbb{R})} = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) \neq 0 \text{ for countably many } x, \text{ and } \sum_x |f(x)|^2 < \infty\}.$$

It is an inner product space with inner product $\langle f, g\rangle := \sum_{x\in\mathbb{R}} f(x)g(x)$, which induces a metric structure and admits a completion. The resulting Hilbert space can't admit any countable orthogonal basis.

¶ Hilbert cube as a "universal" model for all compact metric space.

Roughly speaking, separability means you can use countably many data to "recover" the whole space. Here is an how:

Theorem 1.15. *Any compact metric space (X, d) is (topologically) homeomorphic to a closed subset of the Hilbert cube $([0, 1]^{\mathbb{N}}, d)$.*

Proof. Since X is compact, it is bounded. By scaling the metric d, we may assume $\mathrm{diam}(X) \leqslant 1$. Let $A = \{x_n | n \in \mathbb{N}\}$ be a countable dense subset in X. We define

$$F : X \to [0, 1]^{\mathbb{N}}, \quad x \mapsto (d(x, x_1), d(x, x_2), \cdots , d(x, x_n), \cdots).$$

Then we have:

- F is continuous since $\mathscr{T}_d = \mathscr{T}_{product}$ and each $\pi_n \circ F = d(x, x_n)$ is continuous.
- F is injective: if $F(x) = F(y)$, then $d(x, x_n) = d(y, x_n)$, for all n. Since A is dense, there exists $x_{n_k} \to x$. By continuity of d,
$$d(x, y) = \lim_{k\to\infty} d(x_{n_k}, y) = \lim_{k\to\infty} d(x_{n_k}, x) = 0.$$
- $([0, 1]^{\mathbb{N}}, \mathscr{T}_{product})$ is Hausdorff since it is a metric space.

It follows [see PSet 4-2-1(b)] that the map

$$F : X \to F(X) \subset [0, 1]^{\mathbb{N}}$$

is a homeomorphism. Obviously, $F(X)$ is closed, since it is "a compact subset in a Hausdorff space". □

¶ Other conceptions countability.

Countability is always combined with compactness. For example, we have seen the conception *countably compact* in PSet 4-2-3, and we have seen the conception of σ-compact in Lecture 12. Another countability conception of this flavor is

Definition 1.16. A topological space $(X, \mathscr{T})$ is *Lindelöf* if any open covering $\mathscr{U}$ of X admits a countable sub-covering.

Obviously σ-compact implies Lindelöf, while Lindelöf plus countably compact implies compact. We will leave some properties as exercises.

2. Metrizability

¶ Metrizability.

Definition 2.1. We say a topological space $(X, \mathscr{T})$ is *metrizable* if there exists a metric structure on X so that the metric topology coincides with $\mathscr{T}$.

Example 2.2. $([0,1]^{\mathbb{N}}, \mathscr{T}_{product})$ is metrizable, while $([0,1]^{\mathbb{N}}, \mathscr{T}_{box})$ is NOT metrizable since it is not first countable.

We have seen in Lecture 9 that any metrizable topological space must be first countable, Hausdorff and normal. However, these conditions are not sufficient.

Example 2.3. The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is first countable, Hausdorff, normal but not metrizable:

- We have seen that $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is first countable.
- It is not metrizable since it is separable but not second countable.
- It is Hausdorff, since any $x < y$ can be separated by open sets $[x, y)$ and $[y, y+1)$.
- It remains to show that $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is normal, i.e. disjoint closed sets can be separated by disjoint open sets. So we let A, B be disjoint closed sets. For any $a \in A$, we have $a \in B^c$. Since B^c is open, we can take $\varepsilon_a > 0$ such that $[a, a+\varepsilon_a) \cap B = \emptyset$. Similarly for any $b \in B$ we take $\varepsilon_b > 0$ such that $[b, b+\varepsilon_b) \cap A = \emptyset$. Note that we always has
$$[a, a+\varepsilon_a) \cap [b, b+\varepsilon_b) = \emptyset, \quad \forall a \in A \text{ and } b \in B,$$
otherwise we will have $b \in [a, a+\varepsilon_a)$ or $a \in [b, b+\varepsilon_b)$, which is a contradiction. It follows that
$$U_A := \cup_{a\in A}[a, a+\varepsilon_a) \quad \text{and} \quad U_B := \cup_{b\in B}[b, b+\varepsilon_b)$$
are disjoint open sets separating A and B.

¶ Urysohn's metrization theorem.

Although the metrization problem is subtle in general, it has a very simple answer for second countable spaces. In a paper published posthumously in 1925, Pavel Urysohn[1] proved

Theorem 2.4 (Urysohn's metrization theorem)**.** *A second countable topological space $(X, \mathscr{T})$ is metrizable if and only if it is Hausdorff and normal.*

[1] P. Urysohn, 1898-1924, a famous Soviet mathematician. He was awarded his habilitation with topic "integral equations" at Moscow University in June 1921, and turned to topology after that. In about three years he made a big contribution to dimension theory, and proved many important and fundamental theorems including Urysohn lemma and Urysohn metrization theorem. He died in 1924, at age 26, while swimming off the coast of Brittany, France. He and Pavel Alexandrov formulated the modern definition of compactness in 1923.

We will see next time that any compact Hausdorff space is normal. Thus

Corollary 2.5. *A compact Hausdorff space is metrizable if and only if it is second countable.*

Remark 2.6. One can't replace the assumption "second countable" in Urysohn's metrization theorem by "separable", as can be seen by the counterexample $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$. Also the corollary fails if we replace "second countable" by "separable". (For a counterexample, one can search for the "double arrow space".)

The proof of Urysohn's metrization theorem is similar to that of Theorem 1.15: We will embed $(X, \mathscr{T})$ into the Hilbert cube $[0,1]^{\mathbb{N}}$, so that $(X, \mathscr{T})$ inherits a subspace metric! The only issue is: we don't use countably many points to separate. Instead, we need to construct sufficiently many functions via the countable basis. The main ingredient in the proof is the following very important result, also due to Urysohn, which we will prove next time:

Lemma 2.7 (Urysohn's Lemma)**.** *A topological space $(X, \mathscr{T})$ is normal if and only if for any disjoint closed sets $A, B \subset X$, there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) \supset A, f^{-1}(1) \supset B$.*

¶ Urysohn's metrization theorem: The proof.

Proof of Urysohn's Metrization Theorem. As we mentioned above, we want to construct an "embedding"

$$F : X \to [0,1]^{\mathbb{N}}.$$

We let $\mathcal{B} = \{B_n | n \in \mathbb{N}\}$ be a countable base of $\mathscr{T}$.

$\boxed{\textbf{Step 1.}}$ *Separating sets:* For any $x \in X$ and any open neighborhood U of x, there exists (m, n) s.t. $x \in B_n \subset \overline{B_n} \subset B_m \subset U$.

For any $x \in X$ and any open neighborhood U of x, we first pick B_m so that $x \in B_m \subset U$. Since $\{x\}$ and B_m^c are disjoint closed sets in X, by the definition of normal space, there exists open sets U_1, V_1 such that

$$x \in U_1, \quad B_m^c \subset V_1, \quad U_1 \cap V_1 = \emptyset.$$

Again since $\mathcal{B}$ is a base, there exists $B_n \in \mathcal{B}$ s.t. $x \in B_n \subset U_1$. It follows

$$\overline{B_n} \subset \overline{U_1} \subset V_1^c \subset B_m$$

and thus $x \in B_n \subset \overline{B_n} \subset B_m \subset U$.

$\boxed{\textbf{Step 2.}}$ *Constructing functions:* There exists a sequence of continuous functions $f_1, f_2, \cdots$ such that $\forall x \in X$ and any open $U \ni x, \exists n$ s. t. $f_n(x) = 1, f_n(U^c) = 0$.

For any $(m, n) \in I := \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid \overline{B_n} \subset B_m\} \neq \emptyset$, by applying Urysohn's lemma to the pair of disjoint closed sets $\overline{B_n}$ and B_m^c, we can find a continuous function

$g_{n,m} : X \to [0,1]$ s.t.

$$g_{n,m}(\overline{B_n}) = 1 \quad \text{and} \quad g_{n,m}(B_m^c) = 0.$$

Since I is a countable set, we can renumber $g_{n,m}$'s as $f_1, f_2, f_3, \cdots$. By step 1, the sequence of functions $f_1, f_2, \cdots$ satisfies the demanded property.

Step 3. *Embedding X into $[0,1]^{\mathbb{N}}$.*

Finally we define

$$F : X \to [0,1]^{\mathbb{N}}, \quad x \mapsto (f_1(x), f_2(x), \cdots).$$

We want to show that F is a homeomorphism from X onto $F(X) \subset [0,1]^N$.

Since each f_i is continuous, F is continuous. Moreover, F is injective since for any $x \neq y$, we have $x \in \{y\}^c$ and thus there exists n s.t.

$$f_n(x) = 1 \quad \text{and} \quad f_n(y) = 0.$$

So F is continuous and bijective from X to $F(X)$. To prove F is a homeomorphism onto its image, we only need to prove F is an open map onto its image $F(X)$.

We let $U \subset X$ be open, and $z_0 \in F(U)$. Take $x_0 \in U$ s.t. $z_0 = F(x_0)$. Take n s.t.

$$f_n(x_0) > 0 \quad \text{and} \quad f_n(U^c) = 0.$$

Let $V = \pi_n^{-1}((0, +\infty))$, where π_n is the projection map from $[0,1]^{\mathbb{N}}$ to its n^{th} component. Then V is open in $[0,1]^{\mathbb{N}}$. So

$$W := V \cap F(X)$$

is open in $F(X)$. To prove F is open, it suffices to show $z_0 \in W \subset F(U)$:

- We have $z_0 \in W$ since
$$\pi_n(z_0) = \pi_n(F(x_0)) = f_n(x_0) > 0.$$
- We have $W \subset F(U)$ since for any $z \in W$, there exists x s.t.
$$F(x) = z \quad \text{and} \quad f_n(x) > 0,$$
which implies that $x \in U$ and thus $z \in F(U)$.

Since $F(X)$ is a subset in metric space $[0,1]^{\mathbb{N}}$, it admits a subspace metric whose topology is the same as the subspace topology. Now pull-back the metric to X. It is obvious that the resulting metric topology on X coincides with the original topology on X. $\square$