Topology (H) Lecture 14
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SEPARATION AXIOMS

1. Separation Axioms and Urysohn's lemma

¶ Four separation axioms.

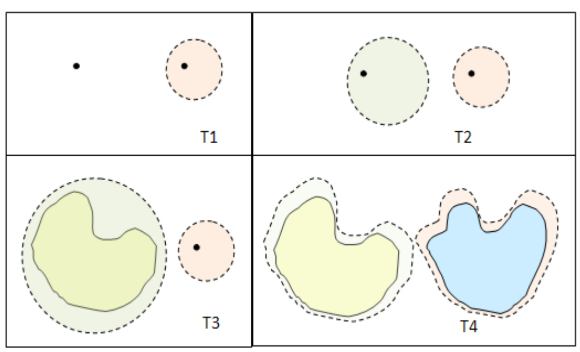
By "separation axioms" we mean properties of topological spaces concerning separating certain disjoint sets via (disjoint) open sets. [Caution: It is very different from the conception *separable* that we learned last time!] There are many different separations axioms [1], four of them are used more often than the others, and we have seen two of them which are most important:

(T1=Frechét) $\forall x_1 \neq x_2 \in X, \exists$ open sets U, V s.t. $x_1 \in U \setminus V$ and $x_2 \in V \setminus U$.

(T2=Hausdorff) $\forall x_1 \neq x_2 \in X, \exists$ open sets U, V s.t. $x_1 \in U, x_2 \in V$ and $U \cap V = \emptyset$.

(T3=Regular) $\forall$ closed sets A and $x \notin A, \exists$ open sets U, V s.t. $A \subset U, x \in V$ and $U \cap V = \emptyset$.

(T4=Normal) $\forall$ closed sets A and B with $A \cap B = \emptyset, \exists$ open sets U, V s.t. $A \subset U, B \subset V$ and $U \cap V = \emptyset$.



[1] In literature there are at least 20 different separation axioms. According to Wikipedia, "the history of the separation axioms in general topology has been convoluted, with many meanings competing for the same terms and many terms competing for the same concept". By the way, the letter "T" comes from the German word "Trennungsaxiom", which means "separation axiom".

Remark 1.1. In different books, "regular", "(T3)", "normal", "(T4)" could have different meanings. For example, in some books "regular" or "(T3)" means "both (T1) and (T3)" in our sense, and "normal" or "(T4)" means "both (T1) and (T4)"; in some other books, "regular" has the same meaning as our's, while "(T3)" means "both (T1) and (T3)" in our sense, and likewise with the meanings of "normal" and "(T4)".

¶ Relations between different separation axioms.

We can also study the relations between these axioms. Obviously we have

- $\boxed{\text{(T2)} \Longrightarrow \text{(T1)}}$,
- $\boxed{\text{(T1)+(T3)} \Longrightarrow \text{(T2)}}$, $\boxed{\text{(T1)+(T4)} \Longrightarrow \text{(T2)}}$, $\boxed{\text{(T1)+(T4)} \Longrightarrow \text{(T3)}}$.

Note: We have

- $\boxed{\text{(T1)} \not\Rightarrow \text{(T2)}}$, $\boxed{\text{(T1)} \not\Rightarrow \text{(T3)}}$, $\boxed{\text{(T1)} \not\Rightarrow \text{(T4)}}$:
  Counterexample: $(\mathbb{R}, \mathscr{T}_{cofinite})$
- $\boxed{\text{(T4)} \not\Rightarrow \text{(T3)}}$, $\boxed{\text{(T4)} \not\Rightarrow \text{(T2)}}$, $\boxed{\text{(T4)} \not\Rightarrow \text{(T1)}}$:
  Counterexample: $(\mathbb{R}, \mathscr{T})$, where $\mathscr{T} = \{(-\infty, a) | a \in \mathbb{R}\}$.
  [It is (T4) because there exists no disjoint closed sets at all!]
- $\boxed{\text{(T3)} \not\Rightarrow \text{(T2)}}$, $\boxed{\text{(T3)} \not\Rightarrow \text{(T1)}}$:
  Counterexample: $(\mathbb{R}, \mathscr{T})$, where $\mathscr{T}$ is generated by
  $$\mathcal{B} = \{[n, n+1) | n \in \mathbb{Z}\}.$$
  In this topology, closed subsets and open subsets are the same.
- $\boxed{\text{(T2)} \not\Rightarrow \text{(T4)}}$, $\boxed{\text{(T2)} \not\Rightarrow \text{(T3)}}$:
  Counterexample: $(\mathbb{R}, \mathscr{T})$ where $\mathscr{T}$ is generated by
  $$\mathcal{S} = \{(a, b) | a, b \in \mathbb{Q}\} \cup \{\mathbb{Q}\}.$$
  $\mathbb{Q}^c$ is closed but it can't be separated from $\{0\}$.
- $\boxed{\text{(T3)} \not\Rightarrow \text{(T4)}}$:
  Counterexample: The Sorgenfrey plane $(\mathbb{R}, \mathscr{T}_{sorgenfrey}) \times (\mathbb{R}, \mathscr{T}_{sorgenfrey})$.
  [Section 31 (Page 152) on Munkres' book.]

¶ Equivalent characterizations.

First we give equivalent characterizations of these axioms.

Proposition 1.2. *Let $(X, \mathscr{T})$ be a topological space.*

(1) $(X, \mathscr{T})$ is (T1) if and only if

$$\boxed{\text{Any single point set } \{x\} \text{ is closed.}}$$

(2) $(X, \mathscr{T})$ is (T2) if and only if

$$\boxed{\text{The diagonal } \Delta = \{(x,x) | x \in X\} \text{ is closed in } X \times X.}$$

(3) $(X,\mathscr{T})$ *is (T3) if and only if*

$$\boxed{\forall x\in U \text{ open, } \exists V \text{ open such that } x\in V\subset \overline{V}\subset U.}$$

(4) $(X,\mathscr{T})$ *is (T4) if and only if*

$$\boxed{\forall \text{ closed } A\subset U \text{ open, } \exists V \text{ open such that } A\subset V\subset \overline{V}\subset U.}$$

Proof. The proof is standard, (1) and (2) follow from definitions, while (3) and (4) follow from open-closed duality:

(1) ($\Rightarrow$) For $\forall y\neq x, \exists U_y\in\mathscr{T}$ such that $x\notin U_y$. So

$$\{x\}^c=\bigcup_{y\neq x}U_y$$

is open, i.e. $\{x\}$ is closed.

($\Leftarrow$) For $\forall x\neq y$, take

$$U=\{y\}^c \quad \text{and} \quad V=\{x\}^c.$$

Then $x\notin V, y\notin U$ and $x\in U, y\in V$.

(2) ($\Rightarrow$) For $\forall x\neq y$, (T2) implies $\exists$ open set $U_x\times V_y$ in $X\times X$ such that

$$(x,y)\in U_x\times V_y \quad \text{and} \quad \Delta\cap(U_x\times V_y)=\emptyset.$$

So Δ^c is open, i.e. Δ is closed.

($\Leftarrow$) For $\forall x\neq y$, i.e. $(x,y)\in\Delta^c$, $\exists$ open sets $U\ni x, V\ni y$ such that

$$(x,y)\in U\times V\subset\Delta^c.$$

It follows $U\cap V=\emptyset$, since if $z\in U\cap V$, then

$$(z,z)\in(U\times V)\cap\Delta=\emptyset.$$

(3) ($\Rightarrow$) Suppose $x\in U$ open, i.e. $x\notin U^c$ closed, then there exists $V_1,V_2\in\mathscr{T}$ s.t.

$$V_1\cap V_2=\emptyset,\ x\in V_1,\ \text{and } U^c\subset V_2.$$

So $x\in V_1\subset\overline{V_1}\subset V_2^c\subset U$.

($\Leftarrow$) Suppose $x\notin A$ closed, i.e. $x\in A^c$ open, then there exists $V\in\mathscr{T}$ such that $x\in V\subset\overline{V}\subset A^c$. It follows

$$V\cap\overline{V}^c=\emptyset,\ x\in V,\ \text{and } A\subset\overline{V}^c.$$

(4) ($\Rightarrow$) Suppose $A\subset U$ open, then $A\cap U^c=\emptyset$. So there exists $V_1,V_2\in\mathscr{T}$ s.t.

$$V_1\cap V_2=\emptyset,\ A\subset V_1,\ \text{and } U^c\subset V_2.$$

So $A\subset V_1\subset\overline{V_1}\subset V_2^c\subset U$.

($\Leftarrow$) Suppose A,B are closed and $A\cap B=\emptyset$. Then $A\subset B^c$ open. So there exists $V\in\mathscr{T}$ such that $A\subset V\subset\overline{V}\subset B^c$. It follows that

$$V\cap\overline{V}^c=\emptyset,\ A\subset V \text{ and } B\subset\overline{V}^c.$$

□

¶ Urysohn Lemma and its proof.

Now we use the equivalent characterization of normal space to prove Urysohn's lemma. Roughly speaking, Urysohn's lemma says

> "Each pair of disjoint closed sets can be separated by open sets if and only if Each pair of disjoint closed sets can be separated by continuous real-valued functions".

Urysohn's lemma is a fundamentally important tool in topology using which one can construct continuous functions with certain properties. For example, we have seen last time how to use Urysohn's lemma to prove Urysohn metrization theorem. Other important applications of Urysohns's lemma include Tietze extension theorem, and embed manifolds into Euclidean spaces. For the case of metric spaces, the proof is very simple because we already have a very nice continuous function – the distance function. However, for general normal spaces, the construction is non-trivial:

Theorem 1.3 (Urysohn's Lemma)**.** *A topological space* $(X, \mathscr{T})$ *is normal if and only if for any pair of disjoint closed subsets* $A, B \subset X$, *there exists a continuous function* $f : X \to [0, 1]$ *such that*

$$A \subset f^{-1}(0) \quad \textit{and} \quad B \subset f^{-1}(1).$$

Proof.

($\Longleftarrow$) This is the easy part: If

$$A \subset f^{-1}(0), B \subset f^{-1}(1)$$

for some continuous function $f : X \to [0, 1]$, then $f^{-1}([0, \frac{1}{3}))$ and $f^{-1}((\frac{2}{3}, 1])$ are disjoint open neighbourhoods of A and B. So $(X, \mathscr{T})$ is normal.

($\Longrightarrow$) (This is the hard part! How do we define a continuous function on a very general topological space? Idea: We can recover a function from its "isoheight lines", i.e. level sets. Of course we don't have the conception of "lines" in general topological space. But: we can first construct enough open sets which will be the "sub-level sets" of the function we are looking for.)

Step 1: Construct "sub-level sets".

Suppose we have a closed set A inside an open set U. We denote $A = A_0, U = U_1$. Since X is normal, we can find open set $U_{\frac{1}{2}}$ and closed set $A_{\frac{1}{2}}$ (which can be taken to be $\overline{U_{\frac{1}{2}}}$ if you want), such that

$$A_0 \subset U_{\frac{1}{2}} \subset A_{\frac{1}{2}} \subset U_1.$$

Repeat the same procedure twice more, we get

$$A_0 \subset U_{\frac{1}{4}} \subset A_{\frac{1}{4}} \subset U_{\frac{1}{2}} \subset A_{\frac{1}{2}} \subset U_{\frac{3}{4}} \subset A_{\frac{3}{4}} \subset U_1.$$

By induction, we can construct, for each dyadic rational number

$$r \in D := \left\{ \frac{m}{2^n} \,\middle|\, n, m \in \mathbb{N}, 1 \leqslant m \leqslant 2^n \right\}$$

an open set U_r and a closed set A_r, such that

①$U_r \subset A_r, \forall r \in D$.

②$A_r \subset U_{r'}, \forall r < r' \in D$.

Step 2: Construct the continuous function from its "sub-level sets".

Now we define

$$f(x) = \inf\{r : x \in U_r\} = \inf\{r : x \in A_r\},$$

where we "define" $\inf \emptyset = 1$. Clearly we have

$$A \subset f^{-1}(0) \quad \text{ and } \quad B = U^c \subset f^{-1}(1).$$

It remains to prove f is continuous. Since

$$\{(-\infty, \alpha) | \alpha \in D\} \cup \{(\alpha, +\infty) | \alpha \in D\}$$

is a sub-basis for the topology of $[0, 1]$, it remains to prove that $f^{-1}((-\infty, \alpha))$ and $f^{-1}((\alpha, +\infty))$ are open for $\forall \alpha \in D$, which follows from the facts

$$f^{-1}((-\infty, \alpha)) = \bigcup_{r<\alpha} U_r \quad \text{ and } \quad f^{-1}((\alpha, +\infty)) = \bigcup_{r>\alpha} A_r^c. \qquad \square$$

Remark 1.4. One may ask: does similar property hold for regular spaces? The answer is NO. In general, we say a topological space is *completely regular* if for any closed subset A and any $x_0 \notin A$, there exists a continuous function $f : [0, 1] \to X$ so that $f(x_0) = 0$ and $f(A) = 1$. One example of regular but not completely regular space, given by J. Thomas in 1969, can be found in *Munkres' book, Problem 11 in Section 33*. Another relatively simple example was constructed by *A. Mysior, Proceeding of the Amer. Math. Soc. Vol. 81 (4), 1981, 652-653.*

¶ F_σ and G_δ sets.

Note that the conclusion of Urysohn's lemma is

$$A \subset f^{-1}(0), \quad B \subset f^{-1}(1).$$

A natural question is:

> Question 1: Under the same assumptions, can we construct continuous function f such that $A = f^{-1}(0), \quad B = f^{-1}(1)$?

The question has a simple answer for metric space, since (c.f. PSet1-2-3) the function $f(x) = d(x, A)/(d(x, A) + d(x, B))$ works. But in general, we need extra assumptions on A and B.

To see this, let's first study the following more fundamental question:

> Question 2: What is the necessary condition for a set $A \subset (X, \mathscr{T})$ so that there exists a continuous function $f : X \to \mathbb{R}$ with $f^{-1}(0) = A$?

Of course we need A to be a closed set. But that is NOT enough: since

$$\{0\} = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}),$$

we must have

$$f^{-1}(0) = \bigcap_{n=1}^{\infty} f^{-1}\left((-\frac{1}{n}, \frac{1}{n})\right).$$

In other words, $f^{-1}(0)$ should be the intersection of countably many open sets in X.

Definition 1.5. Let $(X, \mathscr{T})$ be a topological space, and $A \subset X$.[2]

(1) We say A is a G_δ*-set* if it is a countable intersection of open sets;
(2) We say A is an F_σ*-set* if it is a countable union of closed sets.

Example 1.6. $\mathbb{Q} \subset \mathbb{R}$ is a F_σ-set, $\mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$ is a G_δ-set.

Example 1.7. Any closed subset F in any metric space (X, d) is a G_δ-set since

$$F = \bigcap_{n=1}^{\infty} \left\{ x \;\middle|\; d(x, F) < \frac{1}{n} \right\}.$$

Example 1.8. Consider the space $X = \{0,1\}^{\mathbb{R}}$, equipped with the product topology. Then X is Hausdorff since it is the product of Hausdorff spaces (see today's PSet). It follows that each single point set $\{a\}$ is closed. However, $\{a\}$ is NOT G_δ-set, since any non-empty G_δ-set must contain uncountably many points: Each open set U has only finitely many "non-$\{0, 1\}$" positions (think about a sub-base), which implies that each G_δ-set has only countably many "non-$\{0, 1\}$" positions, and thus contains uncountably many elements. (Note: By Tychonoff, X is compact. We will see soon that any compact Hausdorff space is (T4), so X is a (T4) space.)

So the level set $f^{-1}(0)$ of a continuous function must be a closed G_δ-set. In fact,

Proposition 1.9. *Let X be normal. Then there exists a continuous function $f : X \to [0, 1]$ with $f^{-1}(0) = A$ if and only if A is a closed G_δ set in X.*

Proof. Since A is a G_δ-set, there exists open sets U_n in X such that $A = \bigcap_{n=1}^{\infty} U_n$. By Urysohn's lemma, there exists continuous functions $g_n : X \to [0, 1]$ such that

$$A \subset g_n^{-1}(0), \quad U_n^c \subset g_n^{-1}(1).$$

Now we define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} g_n(x).$$

[2]In mathematics it is very common to use the subscript σ (comes from *somme*, a French word meaning "sum, union") to represent "countable union", and use the subscript δ (comes from *Durchschnitt*, a German word meaning "intersection") to represent "countable intersection". For example, we have seen "σ-algebra", "σ-compact" etc. One can go further and define "$G_{\delta\sigma}$-sets", "δ-ring" etc.

Then f is continuous (since the continuous functions $\sum_{n=1}^{m} \frac{1}{2^n} g_n(x)$ converges to $f(x)$ uniformly). Moreover, we have

$$f^{-1}(0) = A$$

since $A \subset f^{-1}(0)$, and for any $x \notin A$, there exists n such that $x \in U_n^c$, i.e. $g_n(x) = 1$, which implies $f(x) \neq 0$. $\square$

¶ Urysohn's Lemma: a variant.

Given Proposition 1.9, we can easily give a full answer of Question 1, and the trick is almost the same as for metric spaces:

Theorem 1.10 (A variant of Urysohn's lemma)**.** *Let $(X, \mathscr{T})$ be a normal space, and $A, B \subset X$. Then there exists a continuous function $f : X \to [0,1]$ such that*

$$f^{-1}(0) = A, \quad f^{-1}(1) = B$$

if and only if A, B are disjoint closed G_δ-sets in X.

Proof. Obviously if such an f exists, then A, B must be disjoint, closed G_δ-sets.

Conversely let A, B be disjoint, closed G_δ-sets. By Proposition 1.9, we can find continuous functions $f_i : X \to [0,1], i = 1,2$ such that

$$f_1^{-1}(0) = A, f_2^{-1}(0) = B.$$

Since $A \cap B = \emptyset$, we must have $f_1 + f_2 > 0$ on X. So if we define

$$f(x) = \frac{f_1(x)}{f_1(x) + f_2(x)}, \quad \forall x \in X,$$

then $f : X \to [0,1]$ is continuous, and $f^{-1}(0) = A, f^{-1}(1) = B$. $\square$

2. Conditions that guarantee normality.

In this section we study the following question: under which (simpler) assumptions, $(X, \mathscr{T})$ is (T4)? We will prove various results via certain "local-to-global" arguments.

¶ Compactness "enhances" the separation axioms $(T2)$ and $(T3)$.

We first prove

Theorem 2.1. *Any compact Hausdorff space is (T4).*

This is a consequence of the following proposition, which shows how compactness "enhances" the separation axioms (via a simple "local-to-global" argument):

Proposition 2.2. *For topological spaces, we have*

(1) $\boxed{\textit{Compact} + \textit{(T2)} \Longrightarrow \textit{(T3)}}$,

(2) $\boxed{\textit{Compact} + \textit{(T3)} \Longrightarrow \textit{(T4)}}$.

Proof. (1) Let $x \in X, A \subset X$ be closed (and thus compact), and $x \notin A$. Then for any $y \in A$, there exists open sets $U_{x,y} \ni x, V_y \ni y$ such that $U_{x,y} \cap V_y = \emptyset$. By compactness of A, $\exists V_{y_1}, \cdots, V_{y_n}$ cover A. It follows that

$$U := U_{x,y_1} \cap \cdots \cap U_{x,y_n} \quad \text{and} \quad V := V_{y_1} \cup \cdots \cup V_{y_n}$$

are open neighbourhoods of x and A, and $U \cap V = \emptyset$.

(2) Let A, B be disjoint closed subsets. Then for any $x \in A$, there exists open sets $U_x \ni x, V_x \supset B$ such that $U_x \cap V_x = \emptyset$. By compactness, $\exists U_{x_1}, \cdots, U_{x_n}$ cover A. It follows that

$$U := U_{x_1} \cup \cdots \cup U_{x_n} \quad \text{and} \quad V := V_{x_1} \cap \cdots \cap V_{x_n}$$

are open neighbourhood of A and B, and $U \cap V = \emptyset$. □

¶ Countability "enhances" the separation axiom $(T3)$.

In fact, (2) in Proposition 2.2 is a special case of the following proposition, which shows how countability "enhances" the separation axiom (via a more complicated "local-to-global" argument):

Proposition 2.3. *We have* $\boxed{\textit{Lindelöf} + (T3) \Longrightarrow (T4)}$.

As a consequence, $\boxed{(A2) + (T3) \Longrightarrow (T4)}$, $\boxed{\sigma\textit{-compact} + (T3) \Longrightarrow (T4)}$.

Proof. Let A, B be disjoint closed subsets in X. Since Lindeöf is closed hereditary (PSet7-2-3), A, B are also Lindelöf. Since X is (T3), $\forall x \in A$, $\exists$ open set V_x s.t.

$$x \in V_x \subset \overline{V_x} \subset B^c.$$

Since these V_x's cover A which is Lindeöf, we can choose a countable sub-covering $V_1, V_2, \cdots$ that covers A. Similarly one can find countably many open sets $U_1, U_2, \cdots$ that cover B with $U_i \subset \overline{U_i} \subset A^c$. Now let

$$G_n := V_n \setminus (\bigcup_{i=1}^n \overline{U_i}) \quad \text{and} \quad H_n := U_n \setminus (\bigcup_{i=1}^n \overline{V_i}).$$

Then

$$A \subset (\bigcup_{n=1}^\infty V_n) \cap (\bigcap_{i=1}^\infty \overline{U_i}^c) \subset \bigcup_{n=1}^\infty (V_n \cap \bigcap_{i=1}^n \overline{U_i}^c) = \bigcup_{n=1}^\infty G_n$$

and similarly we have

$$B \subset \bigcup_{m=1}^\infty H_m.$$

Finally,

$$(\bigcup_{n=1}^\infty G_n) \cap (\bigcup_{m=1}^\infty H_m) = \emptyset$$

since $G_n \cap H_m = \emptyset$ holds for all n, m. [Check this!] □

Remark 2.4. Comparing Proposition 2.2 and Proposition 2.3, one may ask: do we have "Lindelöf + (T2) $\implies$ (T3)"? The answer is NO. One can find a complicated counterexample in *Steen and Seebach, Counterexamples in Topology*.

¶ Local compactness "enhances" the separation axiom $(T2)$**.**

We can also extend (1) in Proposition 2.2 by replacing "compact" with "locally compact" (again via a different kind of "local-to-global" argument):

Proposition 2.5. *We have* $\boxed{\textit{local compact + (T2)} \implies \textit{(T3)}}$.

Proof. Suppose $x \in U$. By local compactness, we can find a compact set K such that $x \in \mathring{K}$. Since X is (T2), K must be closed and (T2). Since K is compact and (T2), it is (T3). Now let $W = U \cap \mathring{K}$. Then W is an open neighbourhood of x in K. So there exists an open neighbourhood V of x in K such that

$$x \in V \subset \mathrm{Cl}_K(V) \subset W,$$

where we used the notation $\mathrm{Cl}_K(V)$ to represent "the closure of V inside the topological space K", so as to distinguish with $\overline{V}$, the closure of V in X. However, we claim that V is in fact open in X, and $\mathrm{Cl}_K(V)$ coincides with $\overline{V}$:

- Since V is open in K, and $V \subset W$ which is open in X, V has to be open in X.
- Since K is closed in X, and $V \subset K$, we see $\overline{V} \subset K$. It follows $\overline{V} = \mathrm{Cl}_K(V)$.

So we do get an open set V in X such that

$$x \in V \subset \overline{V} \subset W \subset U,$$

which implies X is (T3). □

Remark 2.6. There exist complicated counterexamples showing locally compact Hausdorff space (and thus (T3)) need not be (T4).

¶ Topological manifolds are (T4).

Topological manifolds is a class of very nice and important topological spaces which has applications throughout mathematics.

Definition 2.7. A *topological manifold* is a topological space which is (T2), (A2) and locally Euclidean, i.e. for each $x \in X$ there exists a neighborhood U which is homeomorphic to $\mathbb{R}^n$.

Since any locally Euclidian space is locally compact (c.f. Lecture 12), combining Proposition 2.3 and Proposition 2.5 we get

Proposition 2.8. *Any topological manifold is (T4).*

¶ Paracompactness: definition and examples.

Next we introduce the conception of paracompactness, which, as we will see soon, can be viewed as a mixture of compactness, countability and separability:

Definition 2.9. We say a topological space $(X, \mathscr{T})$ is *paracompact*[3] if any open covering of X admits an open refinement which is locally finite.[4]

Example 2.10. Any compact space is paracompact.

Example 2.11. Any closed subset of a paracompact space is still paracompact.

> Suppose X is paracompact and $A \subset X$ is closed. Let $\mathscr{U}$ be any open covering of A. Let $\mathscr{U}_1 = \mathscr{U} \cup \{A^c\}$. Then it is an open covering of X. By definition, there exists a locally finite refinement $\widetilde{\mathscr{U}_1}$ of $\mathscr{U}_1$. Let
> $$\widetilde{\mathscr{U}} = \{U \in \widetilde{\mathscr{U}_1} \mid U \not\subset A^c\}.$$
> It is easy to see $\widetilde{\mathscr{U}}$ is an open covering of A which is a locally finite refinement of $\mathscr{U}$.

[So paracompactness is closed hereditary. However, it is not hereditary, i.e. an arbitrary subset of a paracompact space could fail to be paracompact.]

Example 2.12. $\mathbb{R}^n$ is paracompact:

> Let $\mathscr{U}$ be any open covering of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, there exits $0 < r_x \leqslant 1$ and $U \in \mathscr{U}$ such that $B(x, r_x) \subset U$. Let
> $$\mathscr{U}_1 = \{B(x, r_x) | x \in \mathbb{R}^n\}.$$
> Then $\mathscr{U}_1$ is a refinement of $\mathscr{U}$. Now any closed ball of the form
> $$\overline{B(a, \sqrt{n})}, \qquad a \in \mathbb{Z}^n$$
> can be covered by finitely many open balls in $\mathscr{U}_1$. Let $\widetilde{\mathscr{U}}$ be the collection of these balls. Then $\widetilde{\mathscr{U}}$ is again an open covering of $\mathbb{R}^n$, is a refinement of $\mathscr{U}$, and is locally finite.

Remark 2.13. More generally, A. Stone[5] proved

> **Theorem 2.14** (Stone)**.** *Any metric space is paracompact.*

Conversely, Smirnov showed that

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[3]The conception of paracompactness was first introduced in 1944 by J. Dieudonné, a French mathematician, one of the founders of *Bourbaki* and drafted much of the Bourbaki series of texts.

[4]Recall

- (Lecture 8) A covering $\mathscr{V}$ is a refinement of $\mathscr{U}$ if $\forall V \in \mathscr{V}$, $\exists U \in \mathscr{U}$ s.t. $V \subset U$.
- (Lecture 7) A family $\{A_\alpha\}$ is locally finite if $\forall x \in X$, there exists open $U_x \ni x$ s.t. $A_\alpha \cap U_x \neq \emptyset$ for only finitely many α's.

[5]Note: A. Stone is the British "Stone", not the American "Stone" who proved Stone-Weierstrass theorem. One need the axiom of choice to prove this theorem.

Theorem 2.15 (Smirnov)**.** *A* locally metrizable *topological space is metrizable if and only if it is paracompact and Hausdorff.*

In fact, Nagata-Smirnov gave a full characterization of metrizability:

Theorem 2.16 (Nagata-Smirnov)**.** *A topological space $(X, \mathscr{T})$ is metrizable if and only if it is (T2), (T3) and admits a σ-locally finite basis.*

where a family of sets $\mathscr{A}$ in X is called *σ-locally finite* if $\mathscr{A} = \cup_n \mathscr{A}_n$, where each $\mathscr{A}_n$ is a locally finite family.

¶ Paracompactness "enhances" the separation axioms $(T2)$ **and** $(T3)$**.**

In general, a paracompact space could fail to be normal. However, just as compactness "enhances" separation axioms $T2$ and $T3$, paracompactness do the same work. The proof is based on another "local-to-global" argument, together with the fact that for a locally finite collection $\{A_\alpha\}$ of subsets, (c.f. Proposition 2.9 in Lecture 7)

$$\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \overline{A}.$$

Proposition 2.17. *We have*

(1) $\boxed{\textit{Paracompact + (T2)} \Longrightarrow \textit{(T3)}}$,

(2) $\boxed{\textit{Paracompact + (T3)} \Longrightarrow \textit{(T4)}}$.

Proof. (1) Suppose B is closed (and thus paracompact) and $x \notin B$. Since X is (T2), $\forall y \in B$, there exist open sets $U_y \ni x, V_y \ni y$ such that $U_y \cap V_y = \emptyset$. Now

$$\mathscr{U}_1 := \{V_y \mid y \in B\}$$

is an open covering of B, which has a locally finite refinement $\widetilde{\mathscr{U}}$. Note: for any $V \in \widetilde{\mathscr{U}}$, there is some $V_y \in \mathscr{U}_1$ such that $V \subset V_y$ and thus $\overline{V} \subset \overline{V_y} \subset U_y^c$. In particular, $x \notin \overline{V}$ for any $V \in \widetilde{\mathscr{U}}$. Let $U = \cup_{V\in\widetilde{\mathscr{U}}} V$. Then U is open and $B \subset U$. Since $\widetilde{\mathscr{U}}$ is locally finite, we have

$$\overline{U} = \overline{\bigcup_{V\in\widetilde{\mathscr{U}}} V} = \bigcup_{V\in\widetilde{\mathscr{U}}} \overline{V}$$

and thus $\overline{U}^c$ is an open neighborhood of x which is disjoint from the open neighborhood U of B. So X is (T3).

(2) Repeat the proof above word by word, with the point x replaced by the closed subset A and "(Ti)" replaced by "(Ti+1)". □

In particular, we get

Theorem 2.18 (Dieudonné)**.** *Every paracompact Hausdorff space is normal.*

¶ A nice refinement for paracompact Hausdorff space.

As an application, we prove

Lemma 2.19. *Let X be paracompact and Hausdorff, and $\mathscr{U} = \{U_\alpha\}$ be an open covering of X. Then there exists a locally finite open refinement $\mathscr{V} = \{V_\alpha\}$* [6] *of $\mathscr{U}$ such that $\overline{V_\alpha} \subset U_\alpha$ for any α.*

Proof. Since X is paracompact and (T2), it is also (T3) and (T4). So if we let

$$\mathscr{A} = \{A \in \mathscr{T} \mid \exists U_\alpha \in \mathscr{U} \text{ s.t. } \overline{A} \subset U_\alpha\},$$

then $\mathscr{A}$ is an open covering of X. Let

$$\mathscr{B} = \{B_\beta \mid \beta \in \Lambda\}$$

be a locally finite open refinement of $\mathscr{A}$, where the index set could be different from that of $\mathscr{A}$. For each β, we choose $\alpha = f(\beta)$ such that

$$\overline{B_\beta} \subset U_{f(\beta)}.$$

Now for each index α of the family $\mathscr{U}$, let

$$V_\alpha = \bigcup_{f(\beta)=\alpha} B_\beta,$$

where we take $V_\alpha = \emptyset$ if no such β exists. By local finiteness of $\mathscr{B}$,

$$\overline{V_\alpha} = \overline{\bigcup_{f(\beta)=\alpha} B_\beta} = \bigcup_{f(\beta)=\alpha} \overline{B_\beta} \subset U_\alpha.$$

It remains to check local finiteness: For any $x \in X$, there exists an open neighborhood U_x of x which intersects with only finitely many B_β's. As a result, U_x only intersects with those α such that $f(\beta) = \alpha$. □

¶ Paracompactness as compactness + countability + separability.

If you stare at the proof of paracompactness for $\mathbb{R}^n$ for a while, you may have seen that the crucial facts we used to construct locally finite open covering are:

- any point x has a compact neighbourhood $\overline{B(x, \sqrt{n})}$. In other words, we are using the *local compactness*!
- the whole space can be covered by countably many such compact balls, that is some kind of global countability [like (A2) or Lindelöf].

A natural question is: does local compactness plus countability like (A2) imply paracompactness? Unfortunately the answer is no in general:

[6] Note: this is a very strong refinement since $\{V_\alpha\}$ and $\{U_\alpha\}$ has the same set of indices.

Example 2.20. Consider $X = \mathbb{R}$ with $\mathscr{T} = \{(-\infty, a) | a \in \mathbb{R}\}$. Then $(X, \mathcal{T})$ is locally compact since any set of the form $(-\infty, x]$ is compact, it is (A2) since

$$\mathscr{B} = \{(-\infty, r) \mid r \in \mathbb{Q}\}$$

is a countable base, but it is NOT paracompact since the open covering

$$\mathscr{U} = \{(-\infty, n) \mid n \in \mathbb{Z}\}$$

has no locally finite refinement.

What is missing in this example? This topology is bad because open sets are too large to separate points (so that there is no local finiteness), i.e. it is NOT Hausdorff! It turns out that local compactness, countability and Hausdorff together implies paracompactness:

Theorem 2.21. *We have* $\boxed{\textit{Lindelöf + locally compact + (T2)} \Longrightarrow \textit{paracompact}}$.
Of course we can replace Lindelöf by a stronger countability like (A2) or σ-compact.

In view of Proposition 2.5, it is enough to prove

Proposition 2.22. *We have* $\boxed{\textit{Lindelöf + (T3)} \Longrightarrow \textit{paracompact}}$.

Proof. Let X be Lindelöf and (T3). Let $\mathscr{U} = \{U_\alpha\}$ be any open covering of X. For any $x \in X$, we choose $\alpha(x)$ such that $x \in U_{\alpha(x)}$. Since X is (T3), we can find open sets V_x and W_x such that

$$x \in V_x \subset \overline{V_x} \subset W_x \subset \overline{W_x} \subset U_{\alpha(x)}.$$

Now $\mathscr{V} = \{V_x\}$ is an open covering of X. Since X is Lindelöf, we can find a countable sub-covering

$$\{V_1, V_2, V_3, \cdots\} \subset \mathscr{V}.$$

We denote $R_1 = W_1$ and define iteratively

$$R_n = W_n \setminus (\overline{V_1} \cup \cdots \cup \overline{V_{n-1}}), \quad n > 1.$$

We claim that $\mathscr{R} = \{R_n\}$ is a locally finite refinement of $\mathscr{U}$:

- By construction, $\mathscr{R}$ is an open refinement of $\mathscr{U}$.
- $\mathscr{R}$ is a covering of X since for any x, if we let n be the least integer such that $x \in W_n$, then $x \notin \overline{V_1} \cup \cdots \cup \overline{V_{n-1}}$ since $\overline{V_i} \subset W_i$. So we must have $x \in R_n$.
- $\mathscr{R}$ is locally finite since for any $x \in X$, we can find n such that $x \in V_n$, and the open neighborhood V_n of x intersects with only finitely many elements in $\mathscr{R}$ since $V_n \cap R_m = \emptyset$ for all $m > n$.

So X is paracompact. □

In particular, we get

Corollary 2.23. *Any topological manifold is paracompact.*

Thus according to Smirnov's theorem, any topological manifold is metrizable.