Topology (H) Lecture 15
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TIETZE EXTENSION THEOREM

1. Tietze Extension Theorem

¶ Tietze Extension Theorem.

Although the function we get via Urysohn's lemma looks too special, they can be used as building blocks to construct more complicated continuous functions with certain properties, as we have seen in the proof of Urysohn's metrization theorem and in the proof of the variant of Urysohn's lemma. In this section we will give another application of Urysohn's lemma, Tietze extension theorem,[1] which can be viewed as a generalization of Urysohn's lemma (although they are in fact equivalent), and thus is directly applicable to more situations. It is one of the most useful theorems in topology.

We start with a trivial definition.

Definition 1.1. Let $A \subset X$ be a subset. We say a map $\tilde{f} : X \to Y$ is an *extension* of a map $f : A \to Y$ if $\tilde{f} = f$ on A.

In analysis, it is always important to extend a given function from a smaller domain to a larger domain, while keeping some properties, e.g. continuity (or smoothness), boundedness. In general, one can't hope to extend *all* continuous functions from A to X if A is not closed, e.g. if $A \subset \mathbb{R}$ is not closed, then there exists a number $a \in A'$ but $a \notin A$, and thus $f(x) := \sin(1/(x-a))$ can't be extended to a continuous function on $\mathbb{R}$. However, if A is closed and X is normal, Tietze extension theorem tells us that any (bounded) continuous function on A admits a continuous extension to X (with the same bounded):

Theorem 1.2 (Tietze Extension Theorem)**.** *A topological space $(X, \mathscr{T})$ is normal if and only if for any closed set $A \subset X$ and any continuous function $f : A \to [-1, 1]$, there exists a continuous function $\widetilde{f} : X \to [-1, 1]$ which is an extension of f.*

Proof. ($\Leftarrow$) Let A, B be disjoint closed sets in X. Then $A \cup B$ is closed in X, and

$$f : A \cup B \to [-1, 1], \quad f(x) = \begin{cases} -1, & x \in A \\ 1, & x \in B \end{cases}$$

[1]According to wikipedia, the theorem was first proved by Brouwer and Lebesgue for the special case of the theorem when X is $\mathbb{R}^n$, and then was extended by Tietze to all metric spaces. The current version for normal space was proved by Urysohn.

is a continuous function on $A \cup B$. By assumption, f can be extended to a continuous function $\widetilde{f} : X \to [-1,1]$ such that $\widetilde{f} = f$ on $A \cup B$. Since $f^{-1}((-\infty, 0))$ and $f^{-1}((0, +\infty))$ are disjoint open neighborhoods of A and B, X is normal.

($\Rightarrow$) **The idea**

Consider the "restriction map"

$$R : \mathcal{C}(X, [-1,1]) \to \mathcal{C}(A, [-1,1]), \quad g \mapsto g|_A,$$

where $\mathcal{C}(X, [-1,1])$ means the space of all continuous maps $g : X \to [-1,1]$. To prove R is surjective, i.e. to solve the equation

$$Rg = f,$$

we apply a standard trick in analysis:

① First find an approximate solution.
② Then iteratively find better and better approximations.
③ Finally prove the sequence of approximate solutions converges to a true solution.

Now we realize the idea.

Step 1 [Construct an approximate solution]

First we approximate the function f by $\bar{f} : A \to \mathbb{R}$, where

$$\bar{f}(x) := \begin{cases} 1/3, & \text{if } f(x) \geqslant 1/3, \\ f(x), & \text{if } |f(x)| \leqslant 1/3, \\ -1/3, & \text{if } f(x) \leqslant -1/3. \end{cases}$$

By construction we have $|f(x) - \bar{f}(x)| \leqslant 2/3$ for any $x \in A$. Then we use Urysohn's lemma to find a continuous function $g : X \to \mathbb{R}$ s.t.

$$Rg \approx \bar{f}.$$

There is a very obvious candidate for such a function g: since

$$A_1 := \{x \in A \,|\, f(x) \geqslant 1/3\} \quad \text{and} \quad B_1 := \{x \in A \,|\, f(x) \leqslant -1/3\}$$

are disjoint closed sets in X, there exists a continuous $g : X \to [-1/3, 1/3]$ s.t.

$$g(x) = 1/3 \text{ on } A_1, \qquad g(x) = -1/3 \text{ on } B_1.$$

It's easy to see that $g(x)$ also satisfies

$$|f(x) - Rg(x)| \leqslant 2/3, \ \forall x \in A.$$

Step 2 [Do iteration]

Write $f = f_1$. According to Step 1, we have obtained a continuous function $g_1 : X \to [-1/3, 1/3]$ s.t.

$$|f_1(x) - Rg_1(x)| \leqslant 2/3, \ \forall x \in A.$$

Repeat Step 1 with f replaced by $f_2 = f_1 - Rg_1$, we can construct a continuous function $g_2 : X \to [-\frac{1}{3}\cdot\frac{2}{3}, \frac{1}{3}\cdot\frac{2}{3}]$ s.t.

$$|f_2(x) - Rg_2(x)| \leqslant (2/3)^2, \ \ \forall x \in A.$$

Iteratively we can find a sequence of continuous functions

$$g_n : X \to \left[-\frac{1}{3}(\frac{2}{3})^{n-1}, \frac{1}{3}(\frac{2}{3})^{n-1}\right]$$

s.t. if we denote $f_{n+1} = f_n - Rg_n$, then

$$|f_n(x) - Rg_n(x)| \leqslant (2/3)^n, \ \ \forall x \in A.$$

Step 3 [Converges to a solution]

Define $\widetilde{f} : X \to [-1, 1]$ by

$$\widetilde{f}(x) := \sum_{n=1}^{\infty} g_n(x).$$

Since each g_n is continuous on X, and

$$|g_n(x)| \leqslant \frac{1}{3}(\frac{2}{3})^{n-1},$$

we see the series converges uniformly and thus $\widetilde{f}$ is continuous on X, and

$$|\widetilde{f}(x)| \leqslant 1, \quad \forall x \in X.$$

Finally for $\forall x \in A$, we have

$$\begin{aligned}\left|f(x) - \sum_{n=1}^{N} g_n(x)\right| &= |f_1 - g_1 - \cdots - g_N| \\ &= |f_2 - g_2 - \cdots - g_N| = \cdots \\ &= |f_N - g_N| \leqslant (2/3)^N.\end{aligned}$$

So $f(x) = \widetilde{f}(x)$ for $x \in A$. □

¶ Extending unbounded functions.

Obviously in the statement of Tietze extension theorem, we can replace the range $[-1, 1]$ by any closed interval $[a, b]$: We only need to compose the functions we get with the linear transform

$$t \mapsto a + t(b - a)$$

and its inverse transform. A not-that-obvious extension: $[-1, 1]$ can be replaced by $\mathbb{R}$.

Theorem 1.3 (Tietze extension theorem for unbounded functions)**.** *Suppose X is normal and $A \subset X$ is closed. Then any continuous function $f : A \to \mathbb{R}$ can be extended to a continuous function $\widetilde{f} : X \to \mathbb{R}$.*

Proof. Composing f with the function $\arctan(x)$, we get a continuous function

$$f_1 := \arctan \circ f : A \to (-\frac{\pi}{2}, \frac{\pi}{2}).$$

By Tietze extension theorem, we can extend f_1 to a continuous function

$$\widetilde{f_1} : X \to [-\frac{\pi}{2}, \frac{\pi}{2}].$$

Let

$$B = \widetilde{f_1}^{-1}(\pm\frac{\pi}{2}).$$

Then B is closed in X, and $B \cap A = \emptyset$. By Urysohn's lemma, there exists a continuous function $g : X \to [0,1]$ s.t.

$$g(A) = 1 \quad \text{and} \quad g(B) = 0.$$

Define

$$h(x) = \widetilde{f_1}(x)g(x).$$

Then h is a continuous function mapping X into $(-\frac{\pi}{2}, \frac{\pi}{2})$. Finally we let

$$\widetilde{f}(x) = \tan h(x).$$

Then $\widetilde{f} : X \to \mathbb{R}$ is continuous, and

$$\widetilde{f}(x) = \tan h(x) = \tan \widetilde{f_1}(x) = \tan f_1(x) = x, \ \forall x \in A.$$ □

¶ Three remarks on extending continuous functions.

We list three remarks on extending continuous functions:

Remark 1.4. Obviously one can also extend continuous vector-valued functions

$$f : A \to [0,1]^n, \quad f : A \to \mathbb{R}^n, \quad \text{or} \quad f : A \to [0,1]^S$$

to continuous vector-valued functions on X, i.e. to

$$\widetilde{f} : X \to [0,1]^n, \quad \widetilde{f} : X \to \mathbb{R}^n, \quad \text{or} \quad \widetilde{f} : X \to [0,1]^S,$$

where S is an arbitrary set. To do so, it suffices to extend each component of f.

Similarly one can extend complex-valued functions while keeping the bound, or extend smooth functions to smooth functions (Whitney extension theorem), or extend Lipschitz functions etc.

Remark 1.5. Instead of assuming X is normal, one may assume X is locally compact Hausdorff. The crucial observation/ingredient is: although locally compact Hausdorff spaces need not be normal (so it is possible that we can't separate disjoint closed sets), we still have a nice separation property, namely Proposition 1.13 in Lecture 12, which allow us to separate a compact set from a closed set! We list the LCH versions of Urysohn's lemma and Tietze extension theorem, and leave their proofs as an exercise:

Theorem 1.6 (Urysohn's Lemma, LCH version). *Let X be a LCH, and K, F be disjoint subsets in X with K compact and F closed. Then there exists a continuous function $f : X \to [0,1]$ such that $f(K) = 1$ and $f(F) = 0$.*

Theorem 1.7 (Tietze extension theorem, LCH version). *Let X be a LCH, and K be a compact subset. Then any continuous function $f : K \to \mathbb{R}$ can be extended to a compactly supported* [2] *continuous function $f : X \to \mathbb{R}$.*

Note that by Theorem 1.6, any LCH is completely regular.

Remark 1.8. On the other hand, for a topological space Y, in general one can't expect to extend all continuous function $f : A \to Y$ to a continuous function $\widetilde{f} : X \to Y$.

- To extend a function $f : \{0,1\} \to Y$ to a continuous map
$$\widetilde{f} : [0,1] \to Y,$$
a necessary condition is: $f(0)$ and $f(1)$ lie in the same *path component* of Y.
- To extend a continuous map $f : S^1 \to Y$ to a continuous map $\widetilde{f} : D \to Y$, where D is the unit disc in the plane, one need to require the image $f(S^1)$ to be *contractible* in Y. In particular, we will see that the identity map
$$f : S^1 \to S^1, x \mapsto x$$
can not be extended to a continuous map $\widetilde{f} : D \to S^1$.

We will study these connectivity phenomena in the second half of this course.

2. Applications of Tietze Extension Theorem

Tietze extension theorem has many applications. For example, in real analysis, Tietze extension theorem was used to produce a sequence of continuous functions that approximates (a.e.) a given measurable function. In what follows we give more applications of Tietze extension theorem.

¶ Application 1: Pseudo-compactness in metric space.

We mentioned in Remark 2.6 in Lecture 9 that a metric space X is compact if and only if it is pseudo-compact, i.e. any continuous function on X is bounded. Now we prove this:

Proposition 2.1. *A metric space (X, d) is compact if and only if any continuous function $f : X \to \mathbb{R}$ is bounded.*

[2] The *support* of a function is defined to be the closed set
$$\mathrm{supp}(f) = \overline{\{x \mid f(x) \neq 0\}},$$
and a function is called *compactly supported* if its support is compact.

Proof. If (X, d) is compact, then by the extreme value theorem, any continuous function $f : X \to \mathbb{R}$ is bounded.

To prove the converse, we argue by contradiction. Suppose (X, d) is non-compact, then there exists $A = \{x_1, x_2, \cdots\}$ such that $A' = \emptyset$. It follows that A is closed and each x_n is isolated in A. So the function

$$f : A \to \mathbb{R}, f(x_n) = n$$

is continuous on A. By Tietze extension theorem, there exists a continuous function $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f} = f$ on A. Obviously $\widetilde{f}$ is an unbounded continuous function on X, a contradiction. ☐

Remark 2.2. What we really proved is: $\boxed{\text{(T4)+limit point compact} \Longrightarrow \text{pseudo-compact}}$.

¶ Application 2: Construct space-filling curves using Cantor set.

Our second application is concerned with the Cantor set C. Recall

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}).$$

As we mentioned at the beginning of Lecture 10, one way to understand C is via the ternary representation of real numbers, i.e. regard C as the image of the map

$$g : \{0, 1\}^{\mathbb{N}} \to [0, 1], \quad a = (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{2}{3^k} a_k.$$

We have checked in PSet 6-1-1(a) that the map g is a homeomorphism from $(\{0, 1\}^{\mathbb{N}}, \mathscr{T}_{product})$ onto the Cantor set C.

On the other hand, the map

$$h : \{0, 1\}^{\mathbb{N}} \to [0, 1]^2, \quad a = (a_1, a_2, \cdots) \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{2^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{2^k}\right)$$

is continuous and surjective:

> To check continuity, one only need to check continuity of each component, which can be done easily via sub-base. The surjectivity is just another way to say that each real number has a binary representation. [Note: h can't be injective, otherwise as a bijective continuous map from a compact space to a Hausdorff space, it will has to be a homeomorphism, which is absurd.]

As a consequence, we get a continuous surjective map

$$h \circ g^{-1} : C \to [0, 1]^2.$$

Since C is closed in $[0, 1]$, Tietze extension theorem indicates that there exists a continuous surjective map

$$f : [0, 1] \to [0, 1]^2.$$

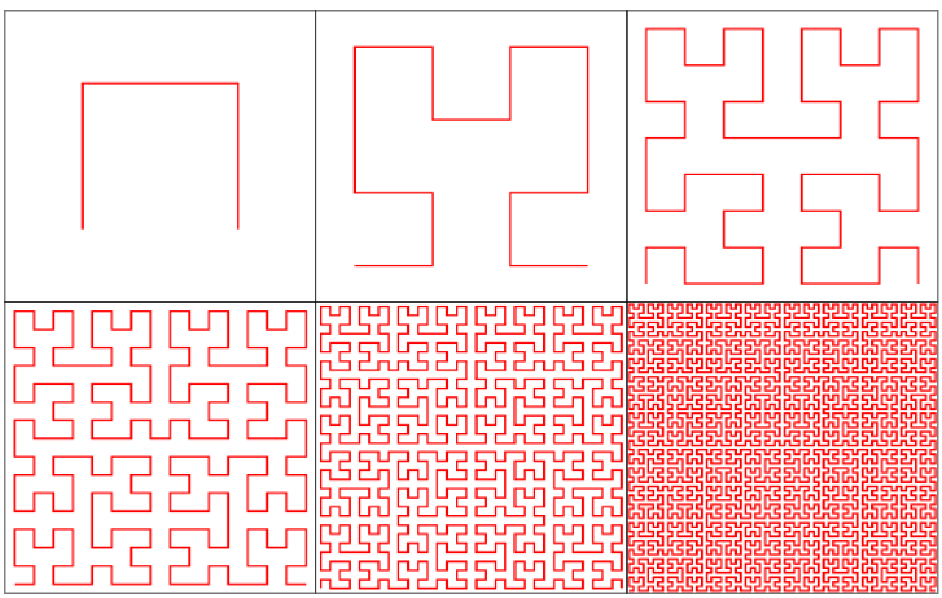
Definition 2.3.

(1) A curve in a topological space X is a continuous map $\gamma : [0,1] toX$.
(2) A *Peano Curve* or a *space-filling curve* is a continuous surjective map from $[0,1]$ to $[0,1]^2$.

So the function f we just constructed is a Peano curve!

Remark 2.4.

(1) By a very similar argument, one can easily construct surjective continuous map
$$f : [0,1] \to [0,1]^n$$
or even surjective continuous map (How? Try to work out this!)
$$f : [0,1] \to [0,1]^{\mathbb{N}}.$$
(2) Of course our argument is an "non-constructive" proof of the existence Peano curve. In literature there are also many "constructive proofs" which iteratively construct such a curve.



(3) The space-filling curves are not just theoretic monsters. They have many practical applications in real life. For example, it is used in storing multidimensional data into computer (which is arranged linearly), e.g. Google maps, so that when you move a little bit on the map, you only move a little bit in the memory, that is why we require continuity of the function.

¶ Application 3: Partition of unity.

One of the most important tools in developing analysis on manifolds is *partition of unity* (subordinate to an open covering):

Definition 2.5. We say a family of functions $\{\rho_\alpha\}$ is a (continuous) *partition of unity* (P.O.U.) if

(1) Each $\rho_\alpha : X \to [0,1]$ is continuous. [Note: ρ_α is defined on the whole of X!]
(2) The family $\{\mathrm{supp}\rho_\alpha\}$ of sets is *locally finite*,
(3) For any $x \in X$, $\sum_\alpha \rho_\alpha(x) = 1$.

We say $\{\rho_\alpha\}$ is a *P.O.U. subordinate to* an open covering $\{U_\alpha\}$ if

(4) For each α, $\mathrm{supp}\rho_\alpha \subset U_\alpha$.

Note that (1) and (2) guarantee the summation in (3) is a continuous function. It turns out that the main ingredient for the existence of P.O.U. is paracompactness:[3]

Theorem 2.6 (Existence of P.O.U.)**.** *Let X be paracompact and Hausdorff. Then for any open covering $\{U_\alpha\}$ of X, there is a partition of unity subordinate to $\{U_\alpha\}$.*

Before we prove the existence of P.O.U. theorem, we first prove a simple version:

Lemma 2.7 (Simple partition of unity)**.** *Let X be normal and $\{K_\alpha\}, \{U_\alpha\}$ be locally finite coverings of X such that for each α, K_α is closed, U_α is open, and $K_\alpha \subset U_\alpha$. Then there exist continuous functions $f_\alpha : X \to [0,1]$ such that*

①$f_\alpha > 0$ *on* K_α.
②$f_\alpha = 0$ *on* U_α^c.
③$\sum_\alpha f_\alpha(x) = 1, \forall x \in X$.

Proof. By Urysohn's lemma, there exist continuous functions $g_\alpha : X \to [0,1]$ such that

$$g_\alpha = 1 \text{ on } K_\alpha, \qquad g_\alpha = 0 \text{ on } U_\alpha^c.$$

Define

$$g(x) = \sum_\alpha g_\alpha(x).$$

Then on open set U_x, g is a finite sum of continuous functions. So g is well-defined and is continuous on each U_x, and hence g is well-defined and is continuous on the whole of X. Moreover,

$$g(x) \geq 1, \quad \forall x$$

since $\bigcup_\alpha K_\alpha = X$. Now we set

$$f_\alpha(x) = \frac{g_\alpha(x)}{g(x)}.$$

It is easy to check that f_α's are what we need. □

[3]Conversely one can prove: for a Hausdorff space, if any open covering admits an P.O.U., then it must be paracompact.

Now we prove Theorem 2.6. The idea is simple: We construct "smaller" locally finite open covering $\{V_\alpha\}$ and "even smaller" closed covering $\{K_\alpha\}$ so that we can apply Lemma 2.7. However, there is still a problem: what we want is $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$, which is stronger than $\rho_\alpha = 0$ on U_α^c in the conclusion of the Lemma. The trick to solve this problem is to construct "even-even-smaller" open covering!

Proof of Theorem 2.6. Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of X. Apply Lemma 2.19 in Lecture 14 three times, we get locally finite open refinement $\mathscr{V} = \{V_\alpha\}$ of $\mathscr{U}$ and locally finite open refinement $\mathscr{W} = \{W_\alpha\}$ of $\mathscr{V}$ and locally finite open refinement $\mathscr{Z} = \{Z_\alpha\}$ of $\mathscr{W}$ (all with the same set of indices) such that

$$\overline{Z_\alpha} \subset W_\alpha \subset \overline{W_\alpha} \subset V_\alpha \subset \overline{V_\alpha} \subset U_\alpha.$$

Now apply Lemma 2.7 for $\overline{Z_\alpha} \subset W_\alpha$, to get continuous functions $\rho_\alpha : X \to [0,1]$ s.t.

- $\rho_\alpha > 0$ on $\overline{Z_\alpha}$,
- $\rho_\alpha = 0$ on W_α^c,
- $\sum_\alpha \rho_\alpha = 1$ on X.

The family $\{\rho_\alpha\}$ is a P.O.U. subordinate to $\{U_\alpha\}$, since

$$\mathrm{supp}\, \rho_\alpha \subset \overline{W_\alpha} \subset U_\alpha,$$

and $\{\mathrm{supp}\, \rho_\alpha\}$ is locally finite since $\mathrm{supp}(\rho_\alpha) \subset V_\alpha$, and $\{V_\alpha\}$ is locally finite.[4] □

Remark 2.8. For LCHs we also have a useful version of partition of unity. In this case we need to assume the space is σ-compact (which is equivalent to Lindelöf for LCH) to guarantee the paracompactness (Theorem 2.21 in Lecture 14), and instead of constructing one continuous function ρ_α for each U_α, we can construct countably many continuous functions $\{\rho_n\}$ so that $\{\mathrm{supp}(\rho_n)\}$ is a refinement of $\{U_\alpha\}$, i.e. for each n, there exists $U_\alpha \in \mathscr{U}$ such that $\mathrm{supp}(\rho_n) \subset U_\alpha$. Moreover, as in the LCH version of Tietze extension theorem, we can require each ρ_n to be compactly supported:

> **Theorem 2.9** (Existence of P.O.U., LCH version)**.** *Let X be locally compact Hausdorff and σ-compact. Then for any open covering $\mathscr{U} = \{U_\alpha\}$ of X. Then there exists a partition of unity $\{\rho_n\}$ such that*
> *(1) each* $\mathrm{supp}(\rho_n)$ *is compact,*
> *(2) for each n, there exists $U_\alpha \in \mathscr{U}$ such that* $\mathrm{supp}(\rho_n) \subset U_\alpha$.

Again we will leave the proof as an exercise. Here is a hint: first, try to construct two locally finite open coverings $\mathscr{V} = \{V_n\}$ and $\mathscr{W} = \{W_n\}$ such that

- $W_n \subset \overline{W_n} \subset V_n \subset \overline{V_n}$, and $\overline{V_n}$ is compact,
- For each n, there exists $U_\alpha \in \mathscr{U}$ such that $\overline{V_n} \subset U_\alpha$.

[4] We may drop $\mathscr{Z}$ and apply Lemma 2.7 to $\overline{W_\alpha} \subset V_\alpha$, and use the fact "$\{V_\alpha\}$ is locally finite $\Longrightarrow$ $\{\overline{V_\alpha}\}$ is locally finite" to prove local finiteness of $\mathrm{supp}(\rho_\alpha)$.

¶ Application 4: Embedding manifolds into $\mathbb{R}^N$.

Theorem 2.6 explains the importance of paracompactness in developing analysis on topological manifolds (which, as we have seen last time, are always paracompact): Since manifolds are locally Euclidean, locally we can pretend that we are working on Euclidian spaces (via coordinates). Then with partition of unity at hand, you can glue these local data to a global one, e.g.

- glue locally defined continuous functions to global continuous functions,
- first define integrals locally and then glue to define global integrals,
- glue locally defined vector fields to global vector fields,
- glue locally defined "inner product structure" to global Riemannian metric

As an application of P.O.U., we prove

Theorem 2.10 (Embedding compact manifolds into Euclidean space)**.** *Any compact topological manifold of dimension n can be embedded into $\mathbb{R}^N$ for some N.*

Proof. Let X be a topological manifold. Then by definition, for each $x \in X$, there exists an open set $U_x \ni x$ and a homeomorphism $\varphi_x : U_x \to V_x$ from U_x to an open set $V_x \subset \mathbb{R}^n$. The triple $\{\varphi_x, U_x, V_x\}$ is called a coordinate chart near x. Since X is compact, we can take a finite covering $\{U_1, \cdots, U_m\}$ of X by such coordinate charts. Let $\{\rho_1, \cdots, \rho_m\}$ be a P.O.U. subordinate to this covering. Define $h_i : X \to \mathbb{R}^n$ by

$$h_i(x) = \begin{cases} \rho_i(x)\varphi_i(x), & x \in U_i \\ (0, \cdots, 0), & x \notin \mathrm{supp}(\rho_i). \end{cases}$$

By pasting lemma (PSet2-2-3(c) with two open sets), each h_i is continuous. Now we let $N = m + mn$ and define $\Phi : X \to \mathbb{R}^N$ by

$$F(x) = (\rho_1, \cdots, \rho_m, h_1, \cdots, h_m).$$

We have

- F is continuous since each component is continuous.
- F injective: If $F(x) = F(y)$, then there exists i such that $\rho_i(x) = \rho_i(y) > 0$, which implies $x, y \in U_i$. It follows that $\varphi_i(x) = \varphi_i(y)$ and thus $x = y$.

Since X is compact and $\mathbb{R}^N$ is Hausdorff, F is a homeomorphism onto its image, i.e. is a topological embedding. □

Remark 2.11. The same theorem holds for non-compact topological manifolds, but the proof is more complicated. c.f. Munkres, Exercise 50.6.