Topology (H) Lecture 16
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CONNECTEDNESS

1. Connectedness: Definitions and examples

¶ Connectedness: The definition.

Connectedness is one of the simplest/most useful topological properties. It is intuitive and is relatively easy to understand, and, it is a powerful tool in proving many well-known results, e.g. the intermediate value theorem (see Lecture 1).

For topological spaces which have simple pictures, it is easy to tell whether the space is connected or not. But for more complicated spaces, it may be more complicated to tell whether the space is connected or not.

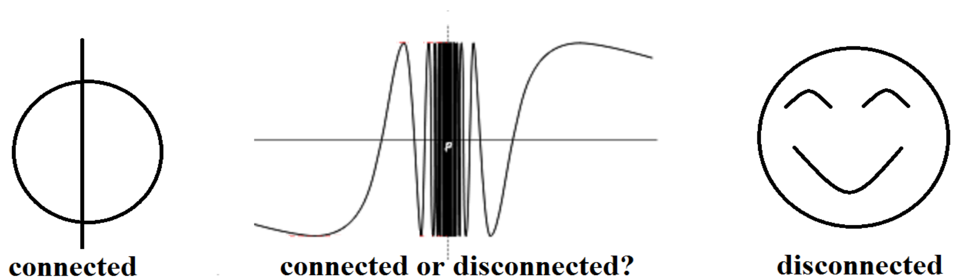

Figure 1. Connected or disconnected

For abstract topological spaces that we don't know how to draw a picture, we also want to ask the question of connectedness. For example, the discrete topological space (with more than one element) should be very disconnected. But, is the Sorgenfrey line connected or disconnected? Is the space of continuous functions on $[0,1]$ connected or disconnected? Of course some of these problems are not quite interesting. However, people do concern on the following problems which arise naturally in analysis: Is $\mathcal{C}(S^1, \mathbb{R}^2)$ connected? Is $\mathcal{C}(S^1, \mathbb{R}^2 \setminus \{0\})$ connected? Is the path space $\{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = \gamma(1)\}$ connected?

So we need a rigorous definition of connectedness (via the collection of open sets). Before we give such a rigorous definition, let's first look at a couple sets in $\mathbb{R}$

$$(a)\ (0,3) \qquad (b)\ (0,1) \cup [2,3) \qquad (c)\ (0,1) \cup (1,3] \qquad (d)\ (0,1] \cup (1,3)$$

Of course (a) is connected, (b) and (c) are disconnected, while (d) is connected! Although (d) looks like a union of two intervals, they are really one interval $(0,3)$ written as a disjoint union of two subsets. The two subsets $(0, 1]$ and $(1, 3)$ are "attached" together at the point 1, which is an element of $(0, 1]$, but sits inside the closure of $(1, 3]$. For the case (c), although the two "components" $(0, 1)$ and $(1, 3]$ sit "next to each other", it is still disconnected because $(0, 1)$ contains no element in the closure of $(1, 3]$, and $(1, 3]$ contains no element in the closure of $(0, 1)$.

This example motivates us to define connectedness. Unlike most other conceptions that you learned, connectedness is defined by its opposite:

Definition 1.1. Let $(X, \mathscr{T})$ be a topological space.

(1) We say X is *disconnected,* if there exists non-empty sets $A, B \subset X$ such that
$$X = A \cup B \quad \text{and } A \cap \overline{B} = \overline{A} \cap B = \emptyset.$$
(2) We say X is *connected* if it is not disconnected.
(3) We say a subset in X is connected/disconnected if it is connected/disconnected with respect to the subspace topology.

Note that by definition, the empty set is connected!

¶ Connectedness: Equivalent characterizations.

The definition above is intuitive but is also a little bit complicated. Fortunately we have several other equivalent ways to describe connectedness.

Proposition 1.2 (Equivalent definitions/characterizatoins of connectedness)**.**

For a topological space X, the following are equivalent:

(1) X is disconnected,
(2) there exist non-empty disjoint open sets $A, B \subset X$ s.t. $X = A \cup B$,
(3) there exist non-empty disjoint closed sets $A, B \subset X$ s.t. $X = A \cup B$,
(4) there exist $A \neq \emptyset, A \neq X$ such that A is both open and closed in X.
(5) there exists a surjective continuous map $f : X \to \{0, 1\}$.

Proof. We have (2) $\iff$ (3) $\iff$ (4) because
$$X = A \cup B \text{ and } A \cap B = \emptyset \iff A^c = B \text{ and } A = B^c.$$
The conclusion (1) $\implies$ (3) follows from
$$A \cap \overline{B} = \emptyset,\ X = A \cup B \implies B = B \cap \overline{B} = X \cap \overline{B} = \overline{B},$$
which implies that B is closed. Similarly A is closed.
Finally to prove (3) $\implies$ (1), we take disjoint closed sets A, B in X such that
$$X = A \cup B.$$
Then $A \cap \overline{B} = A \cap B = \emptyset$ and similarly $\overline{A} \cap B = \emptyset$.

Finally we have (5) $\implies$ (2) trivially, and we have (2) $\iff$ (5) because we can define $f(A) = 0$ and $f(B) = 1$, which is continuous by definition. □

¶ Examples of connected and disconnected spaces.

Example 1.3. $(X, \mathscr{T}_{\text{trivial}})$ is connected, while $(X, \mathscr{T}_{\text{discrete}})$ is disconnected for $|X| \geq 2$.

Example 1.4. $\mathbb{Q} \subset \mathbb{R}$ is disconnected, since

$$\mathbb{Q} = ((-\infty, -\sqrt{2}) \cap \mathbb{Q}) \cup ((-\sqrt{2}, +\infty) \cap \mathbb{Q}).$$

Note that the only connected subsets in $\mathbb{Q}$ are single point sets, since there exist irrational numbers between any two rational numbers. [However, the induced subspace topology on $\mathbb{Q}$ is NOT the discrete topology!]

Definition 1.5. We say a topological space is *totally disconnected* if the only connected subsets are single point sets.

Example 1.6. $\mathbb{Q}$, $\mathbb{Q}^c$, the Cantor set, discrete spaces are all totally disconnected.

Example 1.7. The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$ is totally disconnected: For any subset $A \subset \mathbb{R}$ with at least two elements, say, $a < b$, we take $c \in (a, b)$. By definition, both $(-\infty, c) = \cup_{x<c}[x, c)$ and $[c, +\infty)$ are open in $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$. It follows that $A = A_1 \cup A_2$, where $A_1 = A \cap (-\infty, c)$ and $A_2 = A \cap [c, \infty)$ are both non-empty and open.

Example 1.8. $\mathbb{R}$ is connected (w.r.t. the usual Euclidean topology).

Proof. Suppose $\mathbb{R}$ is disconnected. Then there exists an open set $U \subset \mathbb{R}$ s.t. U^c is also open, and $U \neq \emptyset, U^c \neq \emptyset$. Without loss of generality, we assume that there exist $a < b$ such that $a \in U$ and $b \in U^c$. Let

$$A = \{x \in U \mid x < b\}$$

and let $c = \sup A$. Then

- $c \notin U$: If $c \in U$, then $\exists \varepsilon > 0$ s.t. $b > c + \varepsilon \in U$, a contradiction.
- $c \notin U^c$: If $c \in U^c$, then $\exists\, \varepsilon > 0$ s.t. $(c - \varepsilon, c] \subset U^c$, a contradiction.

So $c \notin U \cup U^c = \mathbb{R}$, a contradiction! □

Remark 1.9. By the same proof, one can show that all intervals

$$(a, b), [a, b], \{a\}, (a, b], [a, b), (a, +\infty), [a, +\infty), (-\infty, b], (-\infty, b), (-\infty, +\infty)$$

are connected. Conversely, by an argument similar to Example 1.4, one can show that these intervals are *the only* non-empty connected subsets of $\mathbb{R}$.

Remark 1.10. In the proof we only used the fact that $\mathbb{R}$ has an order relation $<$ s.t.

(1) Any subset that is bounded above has a least upper bound.
(2) For each pair $x < y, \exists x$ s.t. $x < z < y$.

So the same result holds for any totally ordered set satisfying (1),(2) (called "*Dedekind complete*") equipped with the order topology.

¶ The continuity method.

In particular we see any interval (including $\mathbb{R}$ itself) is connected. This is a simple but very useful fact.

> **The continuity method:**
> To show a family of properties $P(t)$ hold for all $t \in I$, it suffices to check
> (a) $\exists\, t_0 \in I$ s.t. $P(t_0)$ holds.
> (b) $\{t \mid P(t) \text{ holds}\}$ is open in I.
> (c) $\{t \mid P(t) \text{ holds}\}$ is closed in I.
> $\Longrightarrow P(t)$ holds for all $t \in I$.

For example, in proving Calabi conjecture, S.T.Yau need to solve the so-called Monge-Ampére equation $MA(\varphi) = F$. He constructed a family F_t with $F_1 = F$, so that the equation $MA(\varphi) = F_0$ could be solved easily. Then he proved that the set $S = \{t \mid MA(\varphi) = F_t$ admits a solution $\}$ is both open (this step is relatively easy and is also known as the method of continuity) and closed (this step is the hard part where he need to prove various a priori estimates).

Here is a simple example illustrating how to use the continuity method:

Example 1.11. If $f : \mathbb{R} \to \mathbb{R}$ is a real analytic function, and there exists $x_0 \in \mathbb{R}$ s.t.

$$f^{(n)}(x_0) = 0,\ \forall n.$$

Then $f(x) \equiv 0$.

Proof. Let $S = \{x \in \mathbb{R} \mid f^{(n)}(x) = 0, \forall n\}$. Then

- $x_0 \in S \implies S \neq \emptyset$.
- S is open: $x \in S \implies$ any y within the convergence radius of f at x lies in S.
- S closed: $x_n \in S, x_n \to x_0 \implies x_0 \in S$.

Conclusion: $S = \mathbb{R}$, i.e. $f(x) \equiv 0$. □

Remark 1.12. One may regard the continuity method as a "continuous version" of mathematical induction. Sometimes to prove $P(t)$, one need to introduce auxiliary properties $A(t)$ and apply the following variation of continuity method (for example, c.f. T. Tao, *Nonlinear dispersive equations*, §1.3.)

> **Abstract bootstrap principle:**
> To show a family of properties $P(t)$ hold for all $t \in I$, one may introduce auxiliary properties $A(t)$ with the same set of parameters t, and check
> (a) For any t, $A(t)$ implies $P(t)$
> (b) If $P(t_0)$ holds, then $A(t)$ holds for t near t_0
> (c) $\{t \mid P(t) \text{ holds}\}$ is closed in I.
> (d) $\exists\, t_0 \in I$ s.t. $A(t_0)$ holds.
> $\Longrightarrow P(t)$ holds for all $t \in I$.

2. Consequences of connectedness

¶ Generalized intermediate value theorem.

We list a couple properties of connected spaces. The first property is the one we have mentioned in Lecture 1:

Proposition 2.1 (Generalized Intermediate value theorem)**.** *Suppose $f : X \to Y$ is continuous. Then for any connected subset $A \subset X$, the image $f(A) \subset Y$ is connected.*

Proof. By contradiction we suppose $f(A)$ is disconnected. Then there exist non-empty open sets V_1, V_2 in Y with $V_i \cap f(A) \neq \emptyset$ $(i = 1, 2)$ and $V_1 \cap V_2 \cap f(A) = \emptyset$, s.t.

$$f(A) = (V_1 \cap f(A)) \cup (V_2 \cap f(A)).$$

Now let $A_i = f^{-1}(V_i) \cap A$. Then $A_1, A_2 \neq \emptyset, A_1 \cap A_2 = \emptyset$, and

$$f(A) \subset V_1 \cup V_2 \Longrightarrow A = A_1 \cup A_2,$$

which is a contradiction. □

Remark 2.2. Of course it may happen that the image of a disconnected set under a continuous map is connected.

In particular connectedness is a topological property:

Corollary 2.3. *If $f : X \to Y$ is a homeomorphism, then X is connected if and only if Y is connected.*

Since the only connected subsets in $\mathbb{R}$ are intervals, we get

Corollary 2.4 (Intermediate value theorem)**.** *If X is connected, $f : X \to \mathbb{R}$ is continuous, and if there exist $x_1, x_2 \in X$ s.t. $f(x_1) = a < b = f(x_2)$, then for any $a < c < b$, there exists $x \in X$ s.t. $f(x) = c$.*

Proof. $f(X)$ is an interval containing a and b, and thus contains c. □

Another immediate consequence is

Corollary 2.5 (Borsuk-Ulam Theorem, $n = 1$)**.** *For any continuous map $f : S^1 \to \mathbb{R}$, there exists $x_0 \in S^1$ such that $f(x_0) = f(-x_0)$.*

Proof. Define $F : S^1 \to \mathbb{R}$ by $F(x) = f(x) - f(-x)$. Pick any $a \in S^1$. If $F(a) = 0$, we are done. If $F(a) \neq 0$, then both $F(a)$ and $-F(a) = F(-a)$ lie in the image of S^1 under F. Since S^1 is connected, we conclude that 0 is in the image of F. □

¶ The closure.

Next we give several useful criteria for connectedness:

Proposition 2.6. *If $A \subset X$ is connected, $A \subset B \subset \overline{A}$, then B is connected.*

Proof. If B is disconnected, then there exist open sets U_1, U_2 of X s.t. if we let
$$B_1 = U_1 \cap B, \quad B_2 = U_2 \cap B,$$
then $B_1, B_2 \neq \emptyset$ and $B = B_1 \cup B_2$ (so in particular $B \subset U_1 \cup U_2$). We set
$$A_1 = U_1 \cap A, \quad A_2 = U_2 \cap A.$$
Since $A \subset B$, and since A is connected, we conclude that either $A_1 = \emptyset$, or $A_2 = \emptyset$. WLOG, let's assume $A_1 = \emptyset$. Then $A \subset U_1^c$. It follows $\overline{A} \subset U_1^c$, which implies $B \subset U_1^c$. So $B_1 = B \cap U_1 = \emptyset$, contradiction. □

In particular, we get

Corollary 2.7. *If A is connected, so is $\overline{A}$.*

As another consequence, we see that the middle one in Figure 1 is connected:

Corollary 2.8 ("Topologist's sine curve")**.** *For any subset $C \subset \{(0,t) \mid -1 \leq t \leq 1\}$, the set*
$$S = \{(x,y) \mid 0 < x \leq 1, y = \sin\frac{1}{x}\} \cup C \quad \subset \mathbb{R}^2$$
is connected.

¶ The union.

The next proposition, although looks simple, is very useful:

Proposition 2.9. *Let $A_\alpha \subset X$ be a collection of non-empty connected subsets in X, and assume $\cap_\alpha A_\alpha \neq \emptyset$. Then $\cup_\alpha A_\alpha$ is connected.*

Proof. Denote $Y = \cup_\alpha A_\alpha$. Assume $Y = Y_1 \cup Y_2$, where $Y_1 \cap Y_2 = \emptyset$, and
$$Y_1 = Y \cap U_1, \quad Y_2 = Y \cap U_2,$$
where U_1, U_2 are open in X. Take any $x \in \cap_\alpha A_\alpha$. WLOG, assume $x \in Y_1$. For any α, since
$$A_\alpha = (A_\alpha \cap U_1) \cup (A_\alpha \cap U_2),$$
and since $x \in A_\alpha \cap U_1$ which implies $A_\alpha \cap U_1 \neq \emptyset$, we conclude $A_\alpha \cap U_2 = \emptyset$ since A_α is connected. It follows
$$Y_2 = (\bigcup_\alpha A_\alpha) \cap U_2 = \bigcup_\alpha (A_\alpha \cap U_2) = \emptyset.$$
So Y is connected. □

Corollary 2.10. *Suppose $A_1, A_2, \cdots, A_N$ $(N \leq +\infty)$ are connected, and $A_n \cap A_{n+1} \neq \emptyset$ holds for any $n < N$, then $\cup_{n=1}^N A_n$ is connected.*

Proof. By induction and the above proposition, for each n, the set

$$B_n := A_1 \cup \cdots \cup A_n$$

is connected. Now the conclusion follows since $\cap_{n=1}^N B_n \neq \emptyset$. □

¶ The product.

Another consequence of the fact that "the union of connected sets with a common point is connected" is

Corollary 2.11. *If X, Y are connected, so is $X \times Y$.*

Proof. We may assume X, Y are non-empty. Fix $b \in Y$. Then $X \times \{b\}$ is connected since it is the image of a connected set X under a continuous map

$$j_b : X \to X \times Y, \quad x \mapsto (x, b).$$

It follows that for any $x \in X$, the set

$$(\{x\} \times Y) \cup (X \times \{b\})$$

is connected. Moreover, since

$$\bigcap_x (\{x\} \times Y) \cup (X \times \{b\}) \neq \emptyset,$$

we conclude that

$$X \times Y = \bigcup_x ((\{x\} \times Y) \cup (X \times \{b\}))$$

is connected. □

Corollary 2.12. $\mathbb{R}^n, [0,1]^n$ *and* S^n *are connected.*

Proof. For S^n, we can write $S^n = S^n_+ \cup S^n_-$, where $S^n_\pm = S^n \backslash \{0, \cdots, 0, \pm 1\}$ are connected since they are homeomorphic to $\mathbb{R}^n$ via the stereographic projection. □

It turns out that the connectedness is productive:

Proposition 2.13. *If for each $\alpha \in \Lambda$, X_α is connected, then the product space $\prod_\alpha X_\alpha$ is connected with respect to the product topology.*

Proof. For any α, fix an element $a_\alpha \in X_\alpha$. For any finite set of indices $K \subset \Lambda$, by induction the product $\prod_{\alpha \in K} X_\alpha$ is connected. Let

$$X_K = \{(x_\alpha) \mid x_\alpha = a_\alpha \text{ for } \alpha \notin K\}.$$

Then X_K is the image of under the canonical embedding map

$$j_K : \prod_{\alpha \in K} X_\alpha \to \prod_{\alpha \in \Lambda} X_\alpha \simeq \prod_{\alpha \in K} X_\alpha \times \prod_{\alpha \notin K} X_\alpha,^1 \quad (x_\alpha)_{\alpha \in K} \mapsto ((x_\alpha)_{\alpha \in K}, (a_\alpha)_{\alpha \notin K})$$

[1]One should check that the product topology is *commutative and associative*, i.e. if $\Lambda = \cup_\beta \Lambda_\beta$, where $\Lambda_\beta \cap \Lambda_{\beta'} = \emptyset$ fro $\beta \neq \beta'$, then $\prod_{\alpha \in \Lambda} X_\alpha \simeq \prod_\beta \left(\prod_{\alpha \in \Lambda_\beta} X_\alpha\right)$, where each product is endowed with the product topology.

which is continuous, so X_K is connected. Note that by construction, $(a_\alpha) \in \cap_K X_K$. So by Proposition 2.9, the set

$$X := \bigcup_{\text{finite } K\subset\Lambda} X_K$$

is connected. On the other hand, there is no non-empty open set inside

$$X^c = (\bigcup_{\text{finite } K\subset\Lambda} X_K)^c = \bigcap_{\text{finite } K\subset\Lambda} X_K^c.$$

So we conclude

$$\overline{X} = (\mathrm{Int}(X^c))^c = (\emptyset)^c = \prod_\alpha X_\alpha.$$

It follows from Proposition 2.6 that the product space $\prod_\alpha X_\alpha$ is connected. □

Remark 2.14. Conversely if $\prod_\alpha X_\alpha$ is connected, then each X_α is connected since it is the image of $\prod_\alpha X_\alpha$ under the projection map which is continuous.

Remark 2.15. When endowed with the box topology, $\mathcal{M}([0,1],\mathbb{R}) = \mathbb{R}^{[0,1]} = \prod_{\alpha\in[0,1]}\mathbb{R}$ is disconnected, although each component $\mathbb{R}$ is connected. [This is another reason that we prefer the product topology on the product space.]

Reason: Let

$$A = \{f : [0,1] \to \mathbb{R} \mid \exists M \text{ s.t } |f(x)| \le M\},$$
$$B = \{f : [0,1] \to \mathbb{R} \mid \sup_{x\in[0,1]} |f(x)| = +\infty\}.$$

Then $\mathcal{M}([0,1],\mathbb{R}) = A \cup B$, and both A and B are open in box topology:

$$f \in A \implies \{g : [0,1] \to \mathbb{R} \mid g(x) \in (f(x) - 1, f(x) + 1)\} \subset A,$$
$$f \in B \implies \{g : [0,1] \to \mathbb{R} \mid g(x) \in (f(x) - 1, f(x) + 1)\} \subset B.$$

So $\mathcal{M}([0,1],\mathbb{R})$ is disconnected with respect to the box topology.