Topology (H) Lecture 17
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PATH AND PATH CONNECTEDNESS

1. Path and Path-connectedness

¶ Path.

We now turn to a closely related conception: the path connectedness. It is more intuitive, and, as we will see soon, can be extended to define "higher level connectedness" which is described by computable algebraic objects.

Definition 1.1. Let X be a topological space, and $x, y \in X$.

(1) A *path* from x to y is a continuous map $\gamma : [0,1] \to X$ s.t.
$$\gamma(0) = x, \gamma(1) = y.$$
(2) In the case $x = y$, we will call the path a *loop* with base point x.
(3) There is a special path/loop from x to x: the constant path γ_x defined by
$$\gamma_x(t) = x, \quad \forall t \in [0,1].$$

Notations for *path space* and *loop space*:
$$\Omega(X; x_0, x_1) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = x_0, \gamma(1) = x_1\},$$
$$\Omega(X; x_0) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = \gamma(1) = x_0\}.$$

Remark 1.2. So path is a continuous map, not just a "geometric curve". Different parametrizations of the same "geometric pictures" will be regarded as different paths.

It is possible to define some "algebraic operations" on paths:

Definition 1.3. Let X be a topological space.

(1) Given any path γ from x to y in X, we can "reverse" the path γ to get a new path $\bar{\gamma}$ from y to x by letting
$$\bar{\gamma}(t) := \gamma(1-t).$$
[The map $\bar{\gamma}$ is continuous because it is the composition of two continuous maps: the map γ and the map $t \mapsto 1-t$.]
(2) Given two paths, γ_1 from x to y and γ_2 from y to z, we can "connect" the two paths to get a new path $\gamma_1 * \gamma_2$ from x to z by letting (why continuous?)
$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Unfortunately these operations are not "very algebraic". For example, $\gamma * \bar{\gamma}$ is different from $\bar{\gamma} * \gamma$, since the first one is a path from x to x while the second one is a path from y to y. Even in the case $x = y$, they are still different paths since they are double loops going in "opposite orders". Also we want the constant path γ_x to behave like an "identity element", but it is not. We will show how to solve this problem and develop a correct "algebra of paths" next time.

¶ Path connectedness.

Definition 1.4. We say a topological space X is *path-connected* is any two points in X can be connected by a path.

It is easy to prove that path-connectedness is stronger than connectedness:

Proposition 1.5. *If X is path-connected, then X is connected.*

Proof. By contradiction. Suppose there exist nonempty disjoint open sets A and B such that $X = A \cup B$. Take a point x in A, y in B and a path γ from x to y. Then

$$[0,1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$$

is the union of non-empty disjoint open sets, which contradicts with the connectedness of [0, 1]. □

We give some examples:

Example 1.6. Any connected open set $U \subset \mathbb{R}^n$ is path connected.

Reason: [*Connectedness argument*] Fix a point $x \in U$ and consider the set

$$A = \{y \in U \mid \text{ there exists a path from } x \text{ to } y\}.$$

Then

- A is open: For any $y \in A$, we take $\varepsilon > 0$ small enough such that $B(y, \varepsilon) \subset U$. Let γ_1 be a path in U connecting x to y. For any $y_1 \in B(y, \varepsilon)$, let γ_2 be the "line segment path" connecting the center y to y_1, which is given explicitly by

$$\gamma_1(t) = ty_1 + (1-t)y.$$

Then $\gamma * \gamma_1$ is a path from x to y_1. So $y_1 \in A$.
- A is closed: By the same argument one can prove if $y \notin A$, then for any point $y_1 \in B(y, \varepsilon)$, we also have $y_1 \notin A$. So A^c is open, i.e. A is closed.

Since U is connected and since A is non-empty (we always have $x \in A$ since we have the constant curve), we conclude $A = U$. So any point in U can be connected to x by a path. It follows that any two points in U can be connected by a path in U: first connect one point to x, then connect x to the other point.

By the same argument one can prove:

Fact: A topological manifold is path connected if and only if it is connected.

Example 1.7. $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.

> Reason: Since $\mathbb{Q}^2$ is a countable set, for any $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, there exist uncountably many lines l s.t.
>
> $$x \in l \subset \mathbb{R}^2 \setminus \mathbb{Q}^2.$$
>
> Now for $x \neq y \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, pick two such lines, one contains x and the other contains y, such that they are not parallel. Now you travel from x through the first line to the intersection point, then through the second line to y.

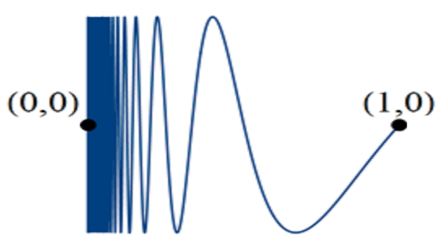
Example 1.8. The topologist's sine curve

$$X = \{(x, \sin\frac{\pi}{x}) \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$

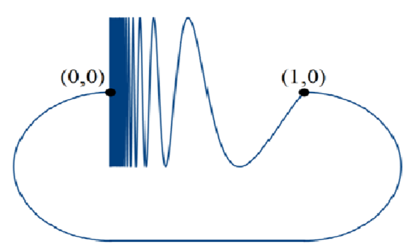
is connected (as we have seen in Lecture 16). But it is NOT path connected.

> Reason: There is no path in X connecting the point $(0,0)$ to $(1,0)$. To see this we suppose $\gamma : [0,1] \to X$ is a path with $\gamma(0) = (0,0)$ and $\gamma(1) = (1,0)$. Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Let $s = \sup\{t \mid \gamma_1(t) = 0\}$. Then $s < 1$, $\gamma_1(s) = 0$ and $\gamma_1(t) > 0$ for all $t > s$. It follows that for $t > s$, $\gamma_2(t) = \sin\frac{\pi}{\gamma_1(t)}$. Now take a decreasing sequence $t_n \to s$ with $\gamma_1(t_n) = \frac{2}{2n+1}$. [The existence of such sequence is guaranteed by the continuity of γ_1.] Then $\gamma_2(t_n) = (-1)^n$ is an oscillating sequence and thus does not converge to $\gamma_2(s)$, a contradiction.

Of course you can make the space path connected by adding a path:



Not path-connected



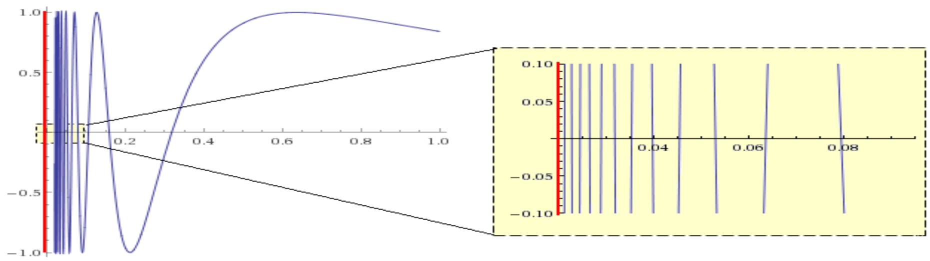
Path connected

From this example we see that in general,

- connected space need not be path connected.
- the closure of path connected subset need not be path connected.

¶ Locally path connectedness.

If you think about the above example carefully, you will find that near any "bad point", say $(0,0)$, inside any small neighborhood you can find infinitely many "vertical curves" that are disconnected:



In other words, "locally near these bad points" the space is not path connected:

Definition 1.9. We say a topological space X is

(1) *locally path connected at x* if for any open neighborhood U of x, there exists an open neighborhood V of x inside U which is path connected.
(2) *locally path connected* if it is locally path connected at any point.

For example, any open set in $\mathbb{R}^n$ (or more generally, in any topological manifold or any locally Euclidian space) is locally path connected. Note that a path connected space may fail to be locally path connected, as shown by the picture "adding a path to topologist's sine curve" above. It turns out that any connected topological space without such bad points is path-connected, and the proof is the same as the proof of path connectedness for connected Euclidian domains above:

Proposition 1.10. *If X is connected and locally path connected, then X is path connected.*

Proof. Fix a point $x \in X$. Consider the set

$$A = \{y \in X \mid y \text{ can be connected by path to } x\}.$$

By locally path connectedness,

- if a point is in A, then a neighborhood of this point is in A,
- if a point is in A^c, then a neighborhood of this point is in A^c.

So $A \neq \emptyset$ is both open and closed. Since X is connected, we must have $A = X$. □

¶ Consequences of path connectedness.

Last time we showed connectedness is preserved under continuous maps, under union with nonempty intersection, and under products. The same properties holds for path connectedness, and the proofs are simpler:

Proposition 1.11. *Let $f : X \to Y$ be continuous. Then for any path connected subset $A \subset X$, the image $f(A)$ is path connected.*

Proof. For any $f(x_1), f(x_2) \in f(A)$, we pick a path $\gamma : [0,1] \to A$ from x_1 to x_2. Then $f \circ \gamma : [0,1] \to f(A)$ is a path from $f(x_1)$ to $f(x_2)$. □

Proposition 1.12. *Let X_α be path connected and $\bigcap_\alpha X_\alpha \neq \emptyset$. Then $\bigcup_\alpha X_\alpha$ is path connected.*

Proof. Take $x_0 \in \bigcap_\alpha X_\alpha$. For any $x_1 \in X_{\alpha_1}$ and $x_2 \in X_{\alpha_2}$, there exist paths γ_1 from x_1 to x_0 and γ_2 from x_0 to x_2. It follows $\gamma_1 * \gamma_2$ is a path from x_1 to x_2. □

Proposition 1.13. *If each X_α is path connected, then the product space $\prod_\alpha X_\alpha$ is also path connected (w.r.t. the product topology).*

Proof. For any $(x_\alpha), (y_\alpha) \in \prod_\alpha X_\alpha$, we pick paths $\gamma_\alpha : [0,1] \to X_\alpha$ from x_α to y_α. Then

$$\gamma : [0,1] \to \prod_\alpha X_\alpha, \quad \gamma(t) = (\gamma_\alpha(t))$$

is continuous and is a path from $\gamma(0) = (x_\alpha)$ to $\gamma(1) = (y_\alpha)$. [Here we used the fact a map to the product is continuous iff each component of the map is continuous.] □

¶ Connected components and path components.

For any topological space X, we can define two equivalence relations,

$$x \sim y \iff \exists \text{ connected subset } A \subset X \text{ s.t. } x, y \in A,$$
$$x \overset{p}{\sim} y \iff \exists \text{ a path in } X \text{ connecting } x \text{ and } y.$$

It is not hard to check that both $\sim$ and $\overset{p}{\sim}$ are equivalence relations.

Definition 1.14. Let X be a topological space.

(1) Each equivalence class for $\sim$ is called a *connected component* of X.
(2) Each equivalence class for $\overset{p}{\sim}$ is called a *path component* of X.

Note that by definition, each connected component can be identified with a maximal (with respect to the set inclusion) connected subset of X, while each path component can be identified with a maximal path connected subset of X.

We have seen in PSet9-1-3 that any connected component of X is a closed subset (which need not be open). However, from the example of topologist's sine curve we see that a path component could be neither closed nor open. On the other hand, in PSet9-1-4 we have seen that if X is locally connected, then any connected component is open (and thus is a clopen). Similarly it is easy to see that if X is locally path connected, then each path component is open (and thus is a clopen).

¶ The space of components.

According to Lecture 6, each equivalence relation on a topological space defines a quotient map and thus a quotient topological space. Thus we get two quotient spaces,

$$\pi_c(X) = X/\sim \qquad \text{and} \qquad \pi_0(X) = X/\overset{p}{\sim}.$$

Note that if X is totally disconnected, then each equivalence class contains only one element and thus $\pi_c(X)$ is X itself. In particular, $X/\sim$ is also totally disconnected. It turns out that the same conclusion holds for any topological space X:

Proposition 1.15. *The quotient space $\pi_c(X)$ (endowed with the quotient topology) is totally disconnected.*

Proof. Let $p : X \to X/\sim$ be the canonical projection map. Suppose $S \subset X/\sim$ is a subset with at least two elements. Then $p^{-1}(S)$ is disconnected. So there exists a nonempty set $A \subsetneq p^{-1}(S)$ which is both open and closed (w.r.t. the subspace topology). Then A must be a union of connected components [since if X_1 is a connected components and $A \cap X_1 \neq \emptyset$, then $X_1 \subset p^{-1}(S)$ and X_1 is also a connected component of $p^{-1}(S)$. It follows that $X_1 = (X_1 \cap A) \cup (X_1 \cap (p^{-1}(S) \setminus A))$ that $X_1 \subset A$.] It follows $A = p^{-1}(p(A))$. By the definition of quotient topology, we conclude that $p(A)$ is both closed and open in $X/\sim$, which implies S is disconnected. □

Remark 1.16. Note that any totally disconnected space is (T1), since by totally disconnectedness, any single point set $\{x\}$ is a connected component and thus is a closed subset, which is equivalent to say that the space is (T1).

For $\pi_0(X)$, again we can think of it as a quotient topological space (with the quotient topology). However, in general the topology could be bad: For example for the topologist's sine curve, the quotient space $\pi_0(X)$ consists of two elements. Let's use "v" to represent the vertical line segment part and use "s" to represent the sine curve part. Then we have

$$\pi_0(\text{topologist's sine curve}) = \{v, s\}, \qquad \mathscr{T}_{quotient} = \{\emptyset, s, \{v, s\}\}.$$

One may guess that the quotient topological space is still totally disconnected, or at least path disconnected, but the truth is: it is path connected:

> Reason: Define $\gamma(t) = s$ for $t < 1/2$ and $\gamma(t) = v$ for $t \geq 1/2$. Then γ is continuous and thus is a path.

Moreover, it is obvious that the quotient space is not even (T1). Since in general the quotient topology on $\pi_0(X)$ is bad and will not give us any useful information, in what follows we would rather forget about the topological structure on the quotient $\pi_0(X)$ and regard $\pi_0(X)$ as a set.

¶ Detour: Functors between categories.

Recall that in Lecture 7 we mentioned the conception of category: a category $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ of objects and a class $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{Mor}(X,Y) \mid X,Y \in \mathrm{Ob}(\mathcal{C})\}$ of morphisms between each pair of objects. Given two categories $\mathcal{C}$ and $\mathcal{D}$, one can relate them by a class of maps called a *functor*:

Definition 1.17. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A (covariant) *functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a mapping that

(1) associates to each object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$,
(2) associates to each morphism $f \in \mathrm{Mor}(X,Y)$ in the category $\mathcal{C}$ a morphism $F(f) \in \mathrm{Mor}(F(X), F(Y))$ in the category $\mathcal{D}$, such that
   - $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for every object $X \in \mathrm{Ob}(\mathcal{C})$,
   - $F(g \circ f) = F(g) \circ F(f)$ [1] for all morphisms $f \in \mathrm{Mor}(X,Y)$ and $g \in \mathrm{Mor}(Y,Z)$ in the category $\mathcal{C}$.

For example,

Example 1.18. Let $\mathcal{TOP}$ be the category of topological spaces, that is:

- Objects are topological spaces,
- Morphisms are continuous maps between topological spaces.

Let $\mathcal{SET}$ be the category of sets, that is

- Objects are sets,
- Morphisms are relations.

Then we can define a "forgetful functor" that maps each topological space to its underlying set, and maps each continuous map to its graph.

Similarly we can let $\mathcal{ALG}$ be the category of (associative) algebras, i.e.

- Objects are (associative) algebras,
- Morphisms are algebra homomorphisms.

Then we can define a (contravariant) functor $\mathcal{C}$ from $\mathcal{TOP}$ to $\mathcal{ALG}$ that assigns to every topological space X the algebra $\mathcal{C}(X,\mathbb{R})$ of all real-valued continuous functions on X, and assigns to each continuous map $f : X \to Y$ the algebra homomorphism $\mathcal{C}(f) : \mathcal{C}(Y,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ defined by $\mathcal{C}(f)(\varphi) := \varphi \circ f$.

¶ Functors π_c and π_0.

Now suppose we have a continuous map $f : X \to Y$ between two topological spaces. According to the generalized intermediate value theorem, f maps each connected component of X *into* a connected component of Y. Thus we get a map

$$\pi_c(f) : X/\sim \to Y/\sim, \qquad [x] \mapsto [f(x)].$$

[1] A *contravariant functor* is a functor that associates to each morphism $f \in \mathrm{Mor}(X,Y)$ a morphism $F(f) \in \mathrm{Mor}(F(Y), F(X))$ such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ and $F(g \circ f) = F(f) \circ F(g)$.

It turns out that π_c has the following nice properties: (proofs left as an exercise.)

Proposition 1.19. *$\pi_c(f)$ is continuous with respect to the quotient topologies. Moreover, $\pi_c(\mathrm{Id}_X) = \mathrm{Id}_{X/\sim}$ and $\pi_c(g \circ f) = \pi_c(g) \circ \pi_c(f)$.*

In other words,

> π_c is a functor from the category of topological spaces, $\mathcal{TOP}$, to the category of totally disconnected topological spaces, $\mathcal{TOP}_{totdis}$:
>
> $$\boxed{\pi_c : \mathcal{TOP} \to \mathcal{TOP}_{totdis}}$$
>
> - $(X, \mathscr{T}) \rightsquigarrow \pi_c(X) = (X/\sim, \mathscr{T}_{quotient})$,
> - $f \in \mathcal{C}(X, Y) \rightsquigarrow \pi_c(f) \in \mathcal{C}(X/\sim, Y/\sim)$

Similarly, since any continuous map $f : X \to Y$ will map a path component in X into a path component in Y, we naturally get a well-defined map (between sets)

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y), \qquad [x] \mapsto [f(x)].$$

We have (proofs left as an exercise.)

Proposition 1.20. *$\pi_0(\mathrm{Id}_X) = \mathrm{Id}_{X/\overset{p}{\sim}}$ and $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.*

In other words,

> π_0 is a functor[2] from the category of topological spaces, $\mathcal{TOP}$, to the category of sets, $\mathcal{SET}$:
>
> $$\boxed{\pi_0 : \mathcal{TOP} \to \mathcal{SET}}$$
>
> - $(X, \mathscr{T}) \rightsquigarrow \pi_0(X) = X/\overset{p}{\sim}$,
> - $f \in \mathcal{C}(X, Y) \rightsquigarrow \pi_0(f) \in \mathcal{M}(X/\overset{p}{\sim}, Y/\overset{p}{\sim})$

Remark 1.21. Of course we lost a lot of information when applying the functor π_c or π_0. But this is exactly the philosophy of algebraic topology: to distinguish topological spaces could be very hard, but very often it would be easier to distinguish objects in simpler categories (like $\mathcal{SET}$, $\mathcal{GROUP}$, or $\mathcal{VECTORSPACE}$). For example, by counting the cardinality of $\pi_c(X)$ or $\pi_0(X)$, we are able to distinguish many topological spaces, e.g. in Lecture 1 we have mentioned

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[2] In fact $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ *is* still continuous w.r.t. the quotient topologies and thus is a functor between the category of topological spaces. Since we prefer to "forget about" the quotient topology on $\pi_0(X)$, this π_0 is indeed the composition of the "topological π_0 functor" with the "forgetful functor".

The second group of figures are topologically different because they have different π_0. For the first group of figures, one can either use π_0 by carefully deleting points in each sides, or by looking at π_1, the fundamental group, which counts the "holes" in the figure.

2. Continuous deformation as a path

¶ Continuous deformation as a continuous family of continuous maps.

In Lecture 1 we have seen the importance of "continuous deformation" in topology via pictures, but without giving it a precise definition. With general topology at hand, we can always give a precise meaning when we talk about "continuous" objects in abstract setting:

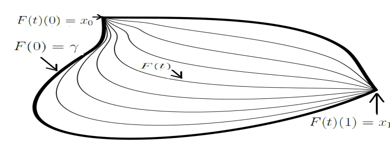
> A *continuous deformation* of an object a in an abstract topological space X is a continuous map $f : [0,1] \to X$ with $f(0) = a$, which may have some extra constraints depending on the problem.

For example, given a path γ from x_0 to x_1 inside a space X, a *continuous deformation of the path* γ with endpoints fixed is a continuous map

$$F : [0,1] \to \Omega(X; x_0, x_1) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = x_0, \gamma(1) = x_1\}$$

which satisfies the conditions

$$F(0) = \gamma,$$
$$F(t)(0) = x_0,$$
$$F(t)(1) = x_1.$$



Wait a minute! We have not specify a topology on the path space $\Omega(X; x_0, x_1)$ yet. Without a topology, it would make no sense to talk about the continuity of F!

Fortunately $\Omega(X; x_0, x_1)$ is a subspace of $\mathcal{C}([0,1], X)$, on which we already have several topologies. In the general case where X is a topological space, from the "convergence point of view" the best topology on $\mathcal{C}([0,1], X)$ is the compact-open topology. [Note: we don't want to use the product (=pointwise convergence) topology here, since we want the limit of a sequence of path (=continuous maps) to be a path!]

More generally, consider a map $f \in \mathcal{C}(X, Y)$, where X, Y are topological spaces. A *continuous deformation of* f over a parameter space T should be a continuous map

$$F : T \to \mathcal{C}(X, Y), \qquad t \mapsto F(t) = f_t \in \mathcal{C}(X, Y)$$

such that $f_{t_0} = f$ for some $t_0 \in T$, where the topology on $\mathcal{C}(X, Y)$ is the *compact-open topology* $\mathscr{T}_{c.o.}$. By definition, this topology is generated by a sub-base

$$\mathscr{S}_{c.o.} = \{S(K, U) \mid K \subset X \text{ is compact and } U \subset Y \text{ is open }\},$$

where

$$S(K,U) = \{f \in \mathcal{C}(X,Y) \mid f(K) \subset U\}.$$

¶ Continuous deformation as one (integrated) continuous map.

Now suppose we have a continuous family (with parameter space T) of maps in $\mathcal{C}(X,Y)$. That is, we have a map $F \in \mathcal{C}(T,\mathcal{C}(X,Y))$. This is still conceptually complicated. But given any such F we can define a much simpler map

$$G \in \mathcal{M}(T \times X, Y), \quad G(t,x) := F(t)(x).$$

It turns out that under mild conditions, F is continuous if and only if G is continuous!

Proposition 2.1. *Suppose X is locally compact Hausdorff, Y, T are arbitrary topological spaces. Consider the correspondence (which is bijection)*

$$\begin{aligned} \mathcal{M}(T,\mathcal{M}(X,Y)) &\longleftrightarrow \mathcal{M}(T \times X, Y) \\ F(t)(x) &\longleftrightarrow G(t,x) := F(t)(x) \end{aligned}$$

Then $F \in \mathcal{C}(T,\mathcal{C}(X,Y))$ if and only if $G \in \mathcal{C}(T \times X, Y)$.

Proof. ($\Longleftarrow$) Suppose $G \in \mathcal{C}(T \times X, Y)$. Then given any $t \in T$, $F(t) : X \to Y$ is continuous since it can be written as the composition of continuous maps

$$X \xrightarrow{j_t} T \times X \xrightarrow{G} Y,$$

where $j_t(x) = (t,x)$ is the "canonical embedding at level t". So F maps T into $\mathcal{C}(X,Y)$. To prove F is continuous as a map from T to $\mathcal{C}(X,Y)$, it is enough to prove

(*) $F^{-1}(S(K,U))$ is open in T for any compact $K \subset X$ and open $U \subset Y$.

Suppose $F(t) \in S(K,U)$. Then by the definition of G, $G(\{t\} \times K) \subset U$, i.e.

$$\{t\} \times K \subset G^{-1}(U).$$

By continuity of G, $G^{-1}(U)$ is open in $T \times X$. Since K is compact, by the tube lemma in Lecture 10, there exist open sets $V \ni t$ in T and $W \supset K$ in X such that

$$V \times W \subset G^{-1}(U).$$

It follows that for any $t \in V$,

$$F(t)(K) \subset G(V \times K) \subset G(V \times W) \subset U.$$

It follows that $V \subset F^{-1}(S(K,U))$, which proves (*).

($\Longrightarrow$) Suppose $F \in \mathcal{C}(T,\mathcal{C}(X,Y))$. To prove G is continuous, we need to show

(**) $G^{-1}(U)$ is open in $T \times X$ for any open set $U \subset Y$.

Suppose $G(t,x) \in U$, i.e. $F(t)(x) \in U$. Then $F(t) \in S(\{x\},U)$. In PSet 7-1-2 we have seen that if X is locally compact and Hausdorff, then

$$S(\{x\},U) = \bigcup_{\text{compact neighborhood } K \text{ of } x} S(K,U).$$

So there exists an open neighbourhood W_x of x s.t. $\overline{W_x}$ is compact, and

$$F(t) \in S(\overline{W_x}, U).$$

Since F is continuous, and $t \in F^{-1}(S(\overline{W_x}, U))$, there must exists an open neighborhood V of t s.t.

$$V \subset F^{-1}(S(\overline{W_x}, U)),$$

i.e. $G(V, \overline{W_x}) \subset U$. It follows

$$V \times W_x \subset V \times \overline{W_x} \subset G^{-1}(U).$$

This completes the proof. □

Remark 2.2. We only used LCH in the proof of the second part. In other words, without LCH condition we can still claim: Any function $G \in \mathcal{C}(T \times X, Y)$ defines a continuous family of continuous maps: $F \in \mathcal{C}(T, \mathcal{C}(X, Y))$.