THE FUNDAMENTAL GROUP

1. THE FUNDAMENTAL GROUP

¶ The fundamental group: definition.

In general neither $\pi(X)$ nor $\pi(X; x_0, y_0)$ are groups. However, if $x_0 = y_0$, then

$$\pi_1(X, x_0) := \pi(X; x_0, x_0)$$

is a group since

- we can multiply any two elements $[\gamma_1]_p, [\gamma_2]_p \in \pi_1(X, x_0)$ to get
  $$[\gamma_1]_p * [\gamma_2]_p = [\gamma_1 * \gamma_2]_p \in \pi_1(X, x_0).$$
- we do have an identity $e = [\gamma_{x_0}]_p \in \pi_1(X, x_0)$, such that for any $[\gamma]_p \in \pi_1(X, x_0)$,
  $$[\gamma]_p * [\gamma_{x_0}]_p = [\gamma]_p = [\gamma_{x_0}]_p * [\gamma]_p.$$
- Any $[\gamma_1]_p \in \pi(X, x_0)$ is invertible with inverse $[\gamma]_p^{-1} := [\bar{\gamma}]_p \in \pi_1(X, x_0)$, and
  $$[\gamma]_p^{-1} * [\gamma]_p = [\gamma_{x_0}]_p = [\gamma]_p * [\gamma]_p^{-1}.$$

So

Proposition 1.1. $\pi_1(X, x_0) = \Omega(X, x_0)/_p \sim$ is a group with respect to the multiplication

$$[\gamma_1]_p * [\gamma_2]_p = [\gamma_1 * \gamma_2]_p, \quad \forall \gamma_1, \gamma_2 \in \Omega(X, x_0)$$

and the inverse

$$[\gamma]_p^{-1} := [\bar{\gamma}]_p, \quad \forall \gamma \in \Omega(X, x_0)$$

Definition 1.2. We call $\pi_1(X, x_0)$ the fundamental group\(^1\) based at $x_0$.

\(^1\)The conception of fundamental group was first introduced by H. Poincaré in 1895 in his seminal paper “Analysis situs”. He then published five supplements to the paper between 1899 and 1904. In these papers Poincaré introduced the concepts of the fundamental group and simplicial homology group, provided an early formulation of the Poincaré duality theorem, introduced the Euler–Poincaré characteristic for chain complexes, and raised several important conjectures including the celebrated Poincaré conjecture. According to Dieudonné, these papers provided the first systematic treatment of topology and revolutionized the subject by using algebraic structures to distinguish between non-homeomorphic topological spaces, founding the field of algebraic topology.
Example 1.3. Let \( X \subseteq \mathbb{R}^n \) (or more generally in a topological vector space) be star-like, and let \( x_0 \in X \) be a “center point” of \( X \), that is, a point so that any point in \( X \) can be connected to \( x_0 \) via the line segment. Last time we constructed a homotopy

\[
F : [0, 1] \times X \to X, \quad (t, x) \mapsto tx_0 + (1 - t)x
\]

between the identity map \( F(0, x) = x \) and constant map \( F(1, x) \equiv x_0 \). Now we let \( \gamma \in \Omega(X, x_0) \) be any loop with base point \( x_0 \). Then

\[
H : [0, 1] \times [0, 1] \to X, \quad (t, s) \mapsto F(t, \gamma(s))
\]

is a path homotopy (here we use the fact \( F(t, x_0) = x_0 \)) between \( \gamma \) and \( \gamma x_0 \). As a result, the fundamental group \( \pi_1(X, x_0) \) contains only one element:

\[
\pi_1(X, x_0) \cong \{e\}.
\]

A natural question: what is \( \pi_1(X, x_1) \) if \( x_1 \) is NOT a “center point”?

**Independence of base point.**

More generally, one may ask: Given any topological space \( X \) and two base points \( x_0, x_1 \in X \), what is the relation between \( \pi_1(X, x_0) \) and \( \pi_1(X, x_1) \)?

**Proposition 1.4.** Suppose \( x_0 \) and \( x_1 \) lie in the same path component of \( X \). Then

\[
\pi_1(X, x_0) \simeq \pi_1(X, x_1).
\]

**Proof.** Let \( \lambda \) be a path connecting \( x_0 \) to \( x_1 \). Then the induced map

\[
\Gamma : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [\gamma]_p \mapsto [\bar{\lambda} \ast \gamma \ast \lambda]_p
\]

is a group homomorphism. To see this we calculate

\[
\Gamma([\gamma_1]_p \ast [\gamma_2]_p) = \Gamma([\gamma_1 \ast \gamma_2]_p) = [\bar{\lambda} \ast \gamma_1 \ast \gamma_2 \ast \lambda]_p
\]

\[
= [\bar{\lambda} \ast \gamma_1 \ast \lambda \ast \gamma_2 \ast \lambda]_p
\]

\[
= [\bar{\lambda} \ast \gamma_1 \ast \lambda]_p \ast [\bar{\lambda} \ast \gamma_2 \ast \lambda]_p
\]

\[
= \Gamma([\gamma_1]_p) \ast \Gamma([\gamma_2]_p).
\]

Note that \( \Gamma \) is invertible and \( \Gamma^{-1} \) is induced by the path \( \bar{\lambda} \):

\[
\Gamma^{-1}([\bar{\gamma}]_p) = [\lambda \ast \bar{\gamma} \ast \lambda]_p = [\bar{\lambda} \ast \gamma \ast \bar{\lambda}]_p.
\]

So \( \Gamma^{-1} \) is also a group homomorphism and thus \( \Gamma \) is a group isomorphism. \( \Box \)

**Remark 1.5.** In studying the fundamental groups of a topological space \( X \), we usually assume \( X \) to be path connected, so that the fundamental groups are independent of the base point. So we may (and will) omit the base point and denote the fundamental group by \( \pi_1(X) \). However, we must point out that in this case, although \( \pi_1(X, x_0) \simeq \pi_1(X, x_1) \) for any \( x_0, x_1 \), the isomorphism may depend on the choice of the path from \( x_0 \) to \( x_1 \). See today’s PSet for more details. So one should keep in mind that unlike \( \pi_1(X, x_0) \), the group \( \pi_1(X) \) is not a concrete group, but only an isomorphism class of groups!
 Simply-connected spaces.

We are always happy to study simplest objects:

**Definition 1.6.** Let $X$ be path-connected. We say $X$ is *simply-connected* if

$$\pi_1(X) = \{e\}.$$ 

**Remark 1.7.** The notion of simple connectedness is not only important in topology, but also plays a fundamental role in complex analysis:

- In fact, the conception of simply-connectedness was first introduced by Bernhard Riemann in his PhD thesis in 1851, in which he proved the famous Riemann mapping theorem: any simply connected domain in $\mathbb{C}$ having at least two boundary points can be conformally mapped onto the unit disk.
- The Cauchy’s integral theorem states that if $U$ is a simply connected domain in $\mathbb{C}$, and $f : U \to \mathbb{C}$ is holomorphic, then the value of every line integral in $U$ with integrand $f$ depends only on the end points of the path.

According to Example 1.3, any star-like domain in $\mathbb{R}^n$ or in any topological vector space is simply connected. This already includes any $\mathbb{R}^n$, any topological vector space, and any convex subsets of them.

**Example 1.8.** For any $n \geq 2$, $S^n$ is simply connected. This is a consequence of the following result (together with the fact that the fundamental group is a topological invariant that we will explain below):

**Proposition 1.9.** If $X = U \cup V$, where

- $U, V \subset X$ are open and simply connected;
- $U \cap V$ is path-connected.

Then $X$ is simply-connected.

**Proof.** Take $x_0 \in U \cap V$ as base point. Let $\gamma : [0, 1] \to X$ be any loop with base point $x_0$, i.e. $\gamma(0) = \gamma(1) = x_0$. By continuity, $\{\gamma^{-1}(U), \gamma^{-1}(V)\}$ is an open cover of $[0, 1]$. By Lebesgue number lemma,

$$\exists 0 < t_0 < t_1 < t_2 < \cdots < t_n = 1 \text{ s.t. } \gamma([t_i, t_{i+1}]) \subset U \text{ or } V.$$ 

Let $\lambda_i$ be a path from $x_0$ to $\gamma(t_i)$ which is contained in $U$ or $V$ or $U \cap V$, depending on whether $\gamma(t_i)$ sits in $U$ or $V$ or $U \cap V$. Let $\gamma_i$ be the (reparametrized) path along $\gamma$

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2Unfortunately Riemann’s original proof is incorrect: his flawed proof depended on the Dirichlet principle (which was named by Riemann himself). Although people could rescue Dirichlet’s principle under certain extra hypothesis as we explained earlier, this method is not enough to prove the theorem in full generality. The first rigorous proof was later given by W. Osgood in 1900.
from $\gamma(t_{i-1})$ to $\gamma(t_i)$. Then

$$\gamma \sim_p \gamma_1 \ast \gamma_2 \ast \cdots \ast \gamma_n$$

$$\sim_p (\gamma_1 \ast \bar{\lambda}_1 \ast (\lambda_1 \ast \gamma_2 \ast \bar{\lambda}_2) \ast \cdots \ast (\lambda_{n-1} \ast \gamma_n)$$

$$\sim_p \gamma_{x_0} \ast \gamma_{x_0} \ast \cdots \ast \gamma_{x_0} = \gamma_{x_0},$$

where in the last step we used the fact that each loop $\lambda_i \ast \gamma_{i+1} \ast \bar{\lambda}_{i+1}$ lies in either $U$ or $V$, which are simply connected.

\[\square\]

**Remark 1.10.** It turns out that the idea of “using open coverings to compute the fundamental group” leads to a very powerful tool in computing fundamental groups: the Van Kampen Theorem. We will study this later.

**Remark 1.11.** A natural question is: What is $\pi_1(S^1)$? For any $n \in \mathbb{Z}$, let

$$\gamma_n : [0, 1] \to S^1, t \mapsto e^{i2\pi nt}.$$ Then $[\gamma_n]_p *[\gamma_m]_p = [\gamma_{n+m}]_p$ since $\gamma_n \ast \gamma_m$ is a reparametrization of $\gamma_{n+m}$. In other words, the map

$$f : \mathbb{Z} \to \pi_1(S^1), \quad n \mapsto [\gamma_n]_p$$

is a group homomorphism. Next time we will show that $f$ is in fact a group isomorphism. [Even in this simple example, the computation of the fundamental group is highly non-trivial and will lead to another powerful method to compute fundamental groups.]

## 2. Various faces of the fundamental group

### \[\downarrow\] \pi_1 as a functor.

First, as we can expect, $\pi_1$ is a functor. To see this, suppose we have a continuous map $f : X \to Y$. Let $y_0 = f(x_0)$. As we have seen last time, if $\gamma_1, \gamma_2 \in \Omega(X, x_0)$ and $\gamma_1 \sim_p \gamma_2$, then $f \circ \gamma_1, f \circ \gamma_2 \in \Omega(Y, y_0)$ and $f \circ \gamma_1 \sim_p f \circ \gamma_2$. In other words, $f$ induces a well-defined map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma]_p \mapsto [f \circ \gamma]_p.$$ Obviously the identity map $\text{Id}_X : X \to X$ induces the identity map

$$(\text{Id}_X)_* = \text{Id}_{\pi_1(X, x_0)} : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

between corresponding groups.

**Proposition 2.1.** We have

1. $f_*$ is a group homomorphism: $f_*([\gamma_1]_p *[\gamma_2]_p) = f_*([\gamma_1]_p) \ast f_*([\gamma_2]_p)$.
2. If $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, then $(g \circ f)_* = g_* \circ f_*$. 
Proof. (1) 
\[
    f_*(\gamma_1_p \ast \gamma_2_p) = f_*(\gamma_1 \ast \gamma_2)_p = [f \circ (\gamma_1 \ast \gamma_2)]_p \\
    = [(f \circ \gamma_1) \ast (f \circ \gamma_2)]_p \\
    = [f \circ \gamma_1]_p \ast [f \circ \gamma_2]_p \\
    = f_*(\gamma_1)_p \ast f_*(\gamma_2)_p.
\]

(2) \((g \circ f)_*([\gamma]_p) = [g \circ f \circ \gamma]_p = g_*(\gamma)_p = f_*(\gamma)_p \ast f_*(\gamma)_p.
\)

□

As a consequence, we see the fundamental group is a topological invariant:

**Corollary 2.2.** If \( f : X \to Y \) is a homeomorphism, then we have a group isomorphism
\[
    \pi_1(X, x_0) \simeq \pi_1(Y, f(x_0)).
\]

To clarify that \( \pi_1 \) is a functor, we need to specify the categories. Note that we have to either restrict ourselves to the category of path connected topological spaces, or the category of “pointed topological spaces” \( \text{PointedTOP} \) with

- objects are “pointed topological spaces” \((X, x_0)\),
- morphisms are “pointed continuous maps” \( f : (X, x_0) \to (Y, y_0) \),

i.e. \( f \in C(X, Y) \) such that \( f(x_0) = y_0 \).

Now one can conclude that \( \pi_1 \) is a functor
\[
    \pi_1 : \text{PointedTOP} \to \text{GROUP}
\]
under which
\[
    (X, x_0) \mapsto \pi_1(X, x_0), \\
    f \in C((X, x_0), (Y, y_0)) \mapsto f_* = \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0).
\]

\[\text{¶ Geometric meaning: deformation-equivalent loops.}\]

There are also many other ways to understand the fundamental groups. For example, since \([0, 1]\) is LCH, we have
\[
    \pi_1(X, x_0) = \Omega(X, x_0)/_p = \pi_0(\Omega(X, x_0)),
\]
where \( \Omega(X, x_0) \) is the set of all loops in \( X \) based at \( x_0 \), equipped with compact-open topology. In other words, \( \pi_1 \) measures the path-connectivity of the loop space:

- A loop corresponds to the trivial element in \( \pi_1(X) \iff \) it can “shrink” to a point.
- Two loops correspond to the same element in \( \pi_1(X) \iff \) one can be “deformed” to the other.

In particular the “larger” the group \( \pi_1(X) \) is, the more “distinguished” loops the space have.

\( \pi_0 \) and \( \pi_1 \) as relative homotopy classes.

Here is another way to relate \( \pi_0 \) and \( \pi_1 \): Instead of regarding \( \pi_1(X, x) \) as the path homotopy classes of loops based at \( x \), we may also regard it as the ordinary homotopy class in the following “pointed sense”:

\[
\pi_1(X, x) = [(S^1, p), (X, x)],
\]

where \( p = (1, 0) \in S^1 = \partial D \subset \mathbb{R}^2 \). In other words, \( \pi_1(X, x) \) is the set of homotopy classes of pointed continuous maps \( f : (S^1, p) \to (X, x) \).

In a very similar way, we can write

\[
\pi_0(X) = \pi_0(X, x) = [(S^0, p), (X, x)],
\]

where \( p = 1 \in S^0 = \{\pm 1\} = \partial I \subset \mathbb{R} \). Let’s explain why:

- \( f \) is a continuous map from \( (S^0, p) \) to \( (X, x) \)
  \( \iff \) \( f \) is map \( f : \{\pm 1\} \to X \) s.t. \( f(1) = x \)
  \( \iff \) choice of a point \( y = f(-1) \in X \).
- Two pointed maps \( f_1, f_2 : (\{\pm 1\}, 1) \to (X, x) \) with
  \( f_1(1) = f_2(1) = x, \quad f_1(-1) = y_1, f_2(-1) = y_2 \)
  are homotopic
  \( \iff \) there exists a continuous deformation \( F : [0, 1] \times (\{\pm 1\}) \to X \) such that
  \( F(t, 1) \equiv x, F(0, -1) = y_1, F(1, -1) = y_2 \)
  \( \iff \exists \) continuous curve \( \gamma(t) = F(t, -1) \) connecting \( y_1 = f_1(-1) \) to \( y_2 = f_2(-1) \)
  \( \iff \) \( y_1 \) and \( y_2 \) lie in the same path component in \( X \).

\(^3\text{Note that in defining the homotopy class for pointed continuous maps } f : (X, x_0) \to (Y, y_0), \text{ we naturally require } F(t, x_0) = y_0 \text{ for all } t, \text{ since we want each } F(t, \cdot) \text{ to be a pointed map. c.f.: the relative homotopy in PSet9-3-3.}\)
Homotopy groups $\pi_n$.

More generally, for $n \geq 2$, we can consider pointed continuous map

$$f : (S^n, p) \to (X, x)$$

where $p = (1, 0, \cdots, 0) \in S^n = \partial B^{n+1} \subset \mathbb{R}^{n+1}$. Two such maps $f_1, f_2$ are homotopic if there exists a continuous map $F : [0,1] \times S^n \to X$ such that

$$F(0, s) = f_1(s), \quad F(1, s) = f_2(s), \quad F(t, p) = x.$$

This again defines an equivalence relation on $C((S^n, p), (X, x))$.

**Definition 2.3.** The set of homotopy classes is denoted by

$$\pi_n(X, x) = [(S^n, p), (X, x)]$$

which is called the $n^{th}$ homotopy group.

We list some basic facts for general homotopy groups, whose proofs will be in a future course called algebraic topology:

1. For any $n \geq 2$, $\pi_n(X, x)$ is a group. Moreover, they are all abelian groups!
   - Geometrically, a homotopy class $[f] = e \in \pi_n(X, x)$ can be “shrunk” to a point.
   - As $\pi_0$ and $\pi_1$, each $\pi_n$ is a homotopy invariant!
   - To define a group operation on $\pi_n(X, x)$, we need an alternative description. Denote $I = [0,1]$, then $I^n \subset \mathbb{R}^n$ is the unit box. We can identify $S^n$ with $I^n/\partial I^n$ and identify $p \in S^n$ with $[\partial I^n] \in I^n/\partial I^n$. So we get another definition of $\pi_n(X, x)$:

$$\pi_n(X, x) = [(I^n, \partial I^n), (X, x)].$$

Now if we have $f_1, f_2 \in C(I^n, X)$ with $f_i(\partial I^n) = x$, then by “shrinking” the first coordinate of $I^n$ we can get

$$\tilde{f}_1 : [0,1/2] \times I^{n-1} \to X, \quad \tilde{f}_2 : [1/2, 1] \times I^{n-1} \to X.$$

They can be “attached” together to get $f_1 \ast f_2 \in C((I^n, \partial I^n), (X, x))$ since $\tilde{f}_1|_{\text{boundary}} \equiv x \equiv \tilde{f}_2|_{\text{boundary}}$. Again one has to pass to homotopy classes to make the operations well-defined and to get a group.

2. Any path from $x_1$ to $x_2$ induces a group isomorphism $\pi_n(X, x_1) \simeq \pi_n(X, x_2)$.
   - Thus we can write $\pi_n(X)$ for path connected $X$.
   - $\pi_1(X, x_0)$ acts by automorphisms on each $\pi_n(X, x_0)$.

3. $\pi_n$ is a functor from $\textbf{PointedTOP}$ to $\textbf{GROUP}$.

**Remark 2.4.** The conception of higher homotopy groups were first defined by Čech in 1932. He proved that these groups are abelian. Then he withdrew his paper under the advice of Alexandroff and Hopf, who thought that these groups could not contain...
any additional information above the homology groups and could not be the right generalizations of the fundamental groups which are in general non-abelian\(^4\). They were wrong! For example, in contrast to the fact \(H_n(S^k) = 0\) for all \(n > k\), the higher homotopy groups \(\pi_n(S^k)\) for \(k > n\) are non-trivial in general, which are surprisingly difficult to compute. Here are some known homotopy groups of spheres (from wikipedia)

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\(S^1\) & \(S^2\) & \(S^3\) & \(S^4\) & \(S^5\) & \(S^6\) & \(S^7\) & \(S^8\) & \(S^9\) & \(S^{10}\) & \(S^{11}\) & \(S^{12}\) & \(S^{13}\) & \(S^{14}\) & \(S^{15}\) \\
\hline
\(H_0\) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\(H_1\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_2\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_3\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_4\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_5\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_6\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_7\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_8\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_9\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_{10}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_{11}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_{12}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_{13}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_{14}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(H_{15}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\end{table}

3. Homotopy equivalence

\emph{Homotopy equivalence.}

Recall that two topological spaces \(X, Y\) are homeomorphic, denoted by \(X \simeq Y\), if
\[\exists f \in C(X, Y), g \in C(Y, X) \text{ s.t. } f \circ g = \text{Id}_Y \text{ and } g \circ f = \text{Id}_X.\]
We have seen that if \(X \simeq Y\), then
\[\pi_1(X, x) \simeq \pi_1(Y, f(x))\]
and in fact \(\pi_n(X, x) \simeq \pi_n(Y, f(x))\) for all \(n\).

On the other hand, we also have examples with \(X \not\simeq Y\) but \(\pi_1(X) \simeq \pi_1(Y)\), e.g.
\[\pi_1(\mathbb{R}^k) = \pi_1(\{p\}) = \{e\}, \forall k,\]
and in fact \(\pi_n(\mathbb{R}^k) = \pi_n(\{p\}) = \{e\}\) for all \(n\) and all \(k\).

Note that “homeomorphism” is a very strong equivalence relation: If \(X \simeq Y\), then \(X, Y\) have the same topological properties (compactness, connectedness, metrizability,

\(^4\)It is known that the first homology group \(H_1(X)\) is the abelian part of \(\pi_1(X)\).
countability, separation properties etc). Now we define a much weaker equivalence relation between topological spaces:

**Definition 3.1.** We say $X$ and $Y$ are *homotopy equivalent*, denoted by $X \sim Y$, if

$$\exists f \in C(X, Y), g \in C(Y, X) \text{ s.t. } f \circ g \sim \text{Id}_Y \text{ and } g \circ f \sim \text{Id}_X.$$  

Such maps $f$ and $g$ are called *homotopy equivalences* between $X$ and $Y$.

**Remarks 3.2.**

(1) In the definition, $f$ and $g$ are the “inverse” to each other only at the homotopy level. They need not be invertible maps.
(2) Obviously if $X \simeq Y$, then $X \sim Y$ (and obviously the converse is not true).
(3) It is easy to check that the homotopy equivalence is an equivalence relation.
(4) Most topological properties (including compactness, (T$_i$), (A$_i$), metrizability etc) are NOT preserved under homotopy equivalence.

**Example 3.3.** We have $S^n \sim \mathbb{R}^{n+1} \setminus \{0\}$.

The homotopy equivalences can be constructed explicitly as follows: Let

$$f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}, x \to x$$

and

$$g : \mathbb{R}^{n+1} \setminus \{0\} \to S^n, x \to x/|x|.$$  

Then obviously $g \circ f = \text{Id}_{S^n}$, and we have $f \circ g \simeq \text{Id}_{\mathbb{R}^{n+1} \setminus \{0\}}$ via

$$F(t, x) = tx + (1 - t)x/|x|.$$  

Here are some examples of homotopy equivalence (but not homeomorphic):
Contractible = homotopy equivalent to a point.

Recall: a topological space $X$ is contractible if $\text{Id}_X : X \rightarrow X$ is null-homotopic. It follows that for any map $f : X \rightarrow \{\text{pt}\}$, $g : \{\text{pt}\} \rightarrow X$, we have

$$f \circ g = \text{Id}_{\{\text{pt}\}} \text{ and } g \circ f = \text{const} \sim \text{Id}_X.$$ 

In other words, any contractible space $X \sim \{\text{pt}\}$. [This includes $\mathbb{R}^k$ for all $k$, and in fact all star-like (including all convex) domains in $\mathbb{R}^k$, and the cone $C(X) = X \times [0, 1]/X \times \{0\}$ of any topological space $X$.]

Conversely, if $X \sim \{\text{pt}\}$, then there exist $f : X \rightarrow \{\text{pt}\}$ and $g : \{\text{pt}\} \rightarrow X$ s.t. $g \circ f \sim \text{Id}_X$. Since $g \circ f$ is a constant map, we conclude that $\text{Id}_X$ is null-homotopic. In other words, we have

**Proposition 3.4.** $X$ is contractible $\iff X \sim \{\text{pt}\}$.

Homotopy equivalence of $\pi_1$.

Although most topological properties are not preserved under homotopy equivalence, it is easy to check that path-connectedness is preserved under homotopy equivalence $\sim$ there is a good chance that the fundamental group (and homotopy groups) will be preserved under homotopy equivalence. This is in fact the case:

**Theorem 3.5.** Let $f : X \sim Y$ be a homotopy equivalence, then $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$ is a group isomorphism.

To prove the theorem, we first prove that homotopic maps induce “almost the same” group homomorphisms. More precisely, suppose $f_1, f_2 \in C(X, Y)$, and $f_1 \sim f_2$. Fix $x_0 \in X$. Let $F : [0, 1] \times X \rightarrow Y$ be a homotopy between $f_1$ and $f_2$. Then

$$\lambda(t) := F(t, x_0), \quad 0 \leq t \leq 1$$

is a path from $y_1 = f_1(x_0)$ to $y_2 = f_2(x_0)$. Let

$$\Lambda : \pi_1(Y, y_1) \rightarrow \pi_1(Y, y_2), \quad [\gamma]_p \mapsto [\lambda \ast \gamma \ast \lambda]_p$$

be the “base-point changing” group isomorphism induced by the path $\lambda$.

**Lemma 3.6.** $(f_2)_* = \Lambda \circ (f_1)_*$ (as group homomorphisms from $\pi_1(X, x_0)$ to $\pi_1(Y, y_2)$).

**Proof.** Take any $[\gamma]_p \in \pi_1(X, x_0)$. We want to construct a path homotopy between $f_2 \circ \gamma$ and $\lambda^{-1} \ast (f_1 \circ \gamma) \ast \lambda$. Let

$$G : [0, 1] \times [0, 1] \rightarrow Y, \quad G(t, s) = F(t, \gamma(s)).$$

Let $\lambda_t$ be the path from $\lambda(t)$ to $\lambda(1) = y_2$ along $\lambda$ (via a linear reparametrization). Then $\lambda_t \ast G(t, s) \ast \lambda_t$ is the path homotopy between $\lambda \ast (f_1 \circ \gamma) \ast \lambda$ and $f_2 \circ \gamma$. \qed

Now we prove the homotopy invariance of the fundamental group:
Proof of Theorem 3.5. From $f \circ g \sim \text{Id}_Y$, we get

$$\Lambda \circ f_* \circ g_* = \text{Id} : \pi_1(Y, y) \to \pi_1(Y, y),$$

so $f_*$ is surjective. Conversely $g \circ f \sim \text{Id}_X$ implies

$$\tilde{\Lambda} \circ g_* \circ f_* = \text{Id} : \pi_1(X, x) \to \pi_1(X, x).$$

It follows $f_*$ is injective. So $f_*$ is a group isomorphism.

It follows that

**Corollary 3.7.** Any contractible space is simply connected.

**Corollary 3.8.** $\pi_1(\mathbb{R}^2 \setminus \{0\}) \simeq \pi_1(S^1)$, and $\pi_1(\mathbb{R}^n \setminus \{0\}) \simeq \{e\}$ for $n \geq 3$. 