Topology (H) Lecture 24
Lecturer: Zuoqin Wang
Time: June 03, 2021

APPLICATIONS OF FUNDAMENTAL GROUPS

1. Applications of $\pi_1(S^1) \simeq \mathbb{Z}$

$\pi_1(S^1)$ is not only the first non-trivial fundamental group, but also the most important fundamental group. We list a number of applications:

¶ Compute the fundamental groups of many simple spaces.

- $\pi_1(S^1 \times \mathbb{R}^n) \simeq \mathbb{Z}$, $\forall n$; $\pi_1(\mathbb{R}^n \setminus \mathbb{R}^{n-2}) \simeq \mathbb{Z}$, $\forall n \geq 2$.
- $\pi_1(\underbrace{S^1 \times \cdots \times S^1}_{r}) \simeq \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r}$; $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{r}) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{r}$.
- $\pi_1(\text{Möbius strip}) \simeq \mathbb{Z}$, $\pi_1(S^2 \vee S^1) \simeq \mathbb{Z}$.
- $\pi_1(S^2 \cup \{(0,0,z) \mid |z| \leq 1\}) \simeq \mathbb{Z}$, $\pi_1(\mathbb{R}^3 \setminus \{(x,y,0) | x^2+y^2=1\}) \simeq \mathbb{Z}$.

We have seen more in PSets and we will see more below and in today's pset.

¶ Distinguishing $\mathbb{R}^2$ with $\mathbb{R}^n$.

Spectral invariants are widely used to distinguish topological spaces. For example,

Proposition 1.1. *For $n \geqslant 3, \mathbb{R}^n$ and $\mathbb{R}^2$ are not homeomorphic.*

Proof. Suppose $\mathbb{R}^n \simeq \mathbb{R}^2$, then $\mathbb{R}^n \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\}$. But

$$\mathbb{R}^n \setminus \{0\} \sim S^{n-1}, \quad \mathbb{R}^n \setminus \{0\} \sim S^1$$

and $\pi_1(S^{n-1}) = \{e\}, \pi_1(S^1) \simeq \mathbb{Z}$, a contradiction! □

Remark 1.2. Note that

- Using path connectivity (i.e. π_0), $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geqslant 2$.
- Using π_1, we just saw $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geqslant 3$.
- In general, using π_k one can prove $\mathbb{R}^k$ is not homeomorphic to $\mathbb{R}^n$ for $n \geqslant k+1$.

¶ Non-retraction.

One can use π_1 to prove that a subspaces is not a retract of the whole space:

Proposition 1.3. *The circle S^1 is not a retract of the closed unit disk D.*

Proof. By PSet 10-1-3(a), if $r : D \to S^1$ is a retraction, then $r_* : \pi_1(D) \to \pi_1(S^1)$ is surjective, which is impossible sine $\pi_1(D) \simeq \{e\}$. □

Similarly one can prove: S^1 (as the equator) is not a retract of S^2, $S^1 \vee S^1$ is not a retract of $S^1 \times S^1$ etc.

One might think that this method works only if the fundamental group of the subspace is larger than the fundamental group of the whole space. This is not the case:

Proposition 1.4. *The boundary circle is not a retract of the Möbius strip.*

Proof. Let M be a Möbius strip, B its boundary circle and C its center circle. Then along the deformation retraction of M to C, B gets mapped onto C with two rotations. It follows that the induced map $\iota_* : \pi_1(C) \simeq \mathbb{Z} \to \pi_1(M) \simeq \mathbb{Z}$ is given by $n \mapsto 2n$. If there is a retraction $r : M \to B$, then $r \circ i = \mathrm{Id}$ implies $r_*(2n) = n$, which can't be a group homomorphism, a contradiction. □

¶ Brouwer's fixed point theorem ($n = 2$).

Theorem 1.5 (Brouwer's fixed point theorem, $n = 2$)**.** *For any continuous map $f : D \to D$, there exists $p \in D$ s.t. $f(p) = p$.*

Proof. Suppose $f(p) \neq p$ for all $p \in D$. Then we can define $g : D \to S^1$ by

$$g(p) = \text{the intersection point of } S^1 \text{ with the ray } \overline{f(p)p}.$$

One can check that the map g is continuous, and $g|_{S^1} = \mathrm{Id}|_{S^1}$. So S^1 is a retract of D, which is a contradiction! □

¶ The fundamental theorem of algebra.

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients.

Theorem 1.6 (The fundamental theorem of algebra)**.** *$\exists z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$.*

Proof. Suppose p has no root. We define a continuous map

$$f : S^1 \to S^1, \quad z \mapsto p(z)/|p(z)|.$$

Then f is null homotopic (to the constant map $f_0(z) \equiv \frac{a_0}{|a_0|}$) via the path homotopy

$$F : [0,1] \times S^1 \to S^1, (t,z) \mapsto p(tz)/|p(tz)|.$$

By Lemma 3.6 in Lecture 19, we conclude that the induced group homomorphism $f_* : \pi_1(S^1) \to \pi_1(S^1)$ is the "zero" homomorphism, i.e. $f_*(m) = 0$ for all $m \in \mathbb{Z} \simeq \pi_1(S^1)$.

On the other hand, f is homotopic to $f_1 : S^1 \to S^1, z \mapsto z^n$ via[1]

$$G : [0,1] \times S^1 \to S^1, \quad (t,z) \mapsto \frac{z^n + a_{n-1}tz^{n-1} + \cdots + a_1 t^{n-1} z + a_0 t^n}{|z^n + a_{n-1}tz^{n-1} + \cdots + a_1 t^{n-1} z + a_0 t^n|}.$$

However, sine $f_1(\gamma_1) = \gamma_n$, the group homomorphism $(f_1)_* : \pi_1(S^1) \to \pi_1(S^1)$ is given by $(f_1)_*(1) = n$ and thus f_* is not the zero homomorphism, a contradiction. □

[1]Note: the numerator is continuous, is obviously nonzero for $t = 0$, and is nonzero for $t \neq 0$ since in this case it equals $t^n p(z/t)$.

¶ Borsuk-Ulam Theorem ($n = 2$).

In Lecture 15 we used the connectedness (i.e. π_0) to prove the following

Theorem 1.7 (Borsuk-Ulam Theorem for $n = 1$)**.** *For any continuous map $f : S^1 \to \mathbb{R}$, there exists $x_0 \in S^1$ such that $f(x_0) = f(-x_0)$.*

Now we use π_1 to prove

Theorem 1.8 (Borsuk-Ulam Theorem for $n = 2$)**.** *For any continuous map $f : S^2 \to \mathbb{R}^2$, there exists $x_0 \in S^2$ such that $f(x_0) = f(-x_0)$.*

Proof. Suppose to the contrary that there exists a continuous map $f : S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ holds for all x. Define a map $g : S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

Then g is continuous and antipodal-preserving, i.e. $g(-x) = -g(x)$.

Let $\iota : S^1 \to S^2$ be the inclusion of equator to the sphere and let $h := g \circ \iota : S^1 \to S^1$. Then for any $[\gamma]_p \in \pi_1(S^1)$,

$$h_*([\gamma]_p) = [h \circ \gamma]_p = [g \circ \iota \circ \gamma]_p = g_*([\iota \circ \gamma]_p) = g_*(e) = e.$$

It follows that h can be lifted to a continuous map $\tilde{h} : S^1 \to \mathbb{R}$. Note that h is antipodal-preserving, i.e. $h(-x) = -h(x)$. So for any $x \in S^1$,

$$p \circ \tilde{h}(-x) = h(-x) = -h(x) = -p \circ \tilde{h}(x).$$

It follows that $\tilde{h}(-x) - \tilde{h}(x) \in \mathbb{Z} + \frac{1}{2}$, and thus by connectedness, there exists $m \in \mathbb{Z}$ such that $\tilde{h}(-x) - \tilde{h}(x) = m + \frac{1}{2}$ for all $x \in S^1$. Now let $F(x) = \tilde{h} \circ p$, where $p : \mathbb{R} \to S^1$ is the canonical covering map. Then for any $x \in \mathbb{R}$,

$$F(x + \frac{1}{2}) = \tilde{h} \circ p(x + \frac{1}{2}) = \tilde{h}(-p(x)) = \tilde{h}(p(x)) + m + \frac{1}{2} = F(x) + m + \frac{1}{2}.$$

As a consequence, we get

$$F(1) = F(\frac{1}{2}) + m + \frac{1}{2} = F(0) + 2m + 1 \neq F(0),$$

which is a contradiction since $p(0) = p(1)$. □

By using higher invariants like homotopy groups or homology groups, one can prove

Theorem 1.9 (Borsuk-Ulam Theorem)**.** *For any continuous map $f : S^n \to \mathbb{R}^n$, there exists $x_0 \in S^2$ such that $f(x_0) = f(-x_0)$.*

Corollary 1.10. *No subset in $\mathbb{R}^n$ is homeomorphic to S^n.*

Proof. By Borsuk-Ulam, there exists no injective continuous map $f : S^n \to \mathbb{R}^n$. □

¶ Borsuk-Ulam and Lusternik-Schnirelmann for all n.

We say a map defined on S^n is *antipodal-preserving* if $f(-x)=-f(x)$ for all $x \in S^n$. It turns out that Borsuk-Ulam theorem has many equivalent forms. (The proof will be left as an exercise.)

Proposition 1.11. *The following statements are equivalent (and thus are all true):*

1. *For any continuous $f : S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ s.t. $f(x_0) = f(-x_0)$.*
2. *There does not exist an antipodal-preserving continuous map $f : S^n \to S^{n-1}$.*
3. *For any antipodal-preserving continuous map $f : S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ s.t. $f(x_0) = 0$.*
4. *There does not exist continuous map $f : B^n \to S^{n-1}$ s.t. the restriction to the boundary of f, $f|_{S^{n-1}} : S^{n-1} \to S^{n-1}$, is antipodal-preserving.*
5. *Let $F_1, F_2, \cdots, F_{n+1}$ be a covering of S^n by closed sets, then there exists $1 \leqslant i \leqslant n+1$ s.t. $F_i \cap (-F_i) \neq \emptyset$.*
6. *Let $U_1, U_2, \cdots, U_{n+1}$ be a covering of S^n by open sets, then there exists $1 \leqslant i \leqslant n+1$ s.t. $U_i \cap (-U_i) \neq \emptyset$.*

Remark 1.12. The theorem was proved by Borsuk in 1933. The statement (1), i.e. the Borsuk-Ulam theorem, was first conjectured by Ulam. The assertion (5)-(6) was first proved by Lusternik and Schnirelmann using combinatorial method, and is frequently called the Lusternik-Schnirelmann-Borsuk theorem.

¶ Ham-Sandwich Theorem.

Finally we prove the Ham-Sandwich Theorem that we mentioned in Lecture 1: [But unfortunately I can only prove the version for $n = 2$, since I only established the Borsuk-Ulam theorem for $n = 2$. You can prove it for general n in a similar way as long as you assume Borsuk-Ulam for general n.]

Corollary 1.13 (Ham-Sandwich Theorem, $n = 2$)**.** *Given two (nice, at least measurable) bounded sets in $\mathbb{R}^2$, there exists a line that cut each of the two sets into halves.*

Sketch of proof. For any $u = (u_0, u_1, u_2) \in S^2$, if u_1 or $u_2 \neq 0$, then denote

$$L_u = \{(x_1, x_2) | u_1 x_1 + u_2 x_2 < u_0\};$$

if $u = (\pm 1, 0, 0)$, then denote

$$L_{(1,0,0)} = \mathbb{R}^2, \quad L_{(-1,0,0)} = \emptyset.$$

Denote the two sets by A and B. One can check the map

$$f : S^2 \to \mathbb{R}^2, \qquad u \mapsto (\text{area}(A \cap L_u), \text{area}(B \cap L_u))$$

is continuous. So there exists $u^0 = (u_0^0, u_1^0, u_2^0) \in S^2$ s.t. $f(u) = f(-u)$, which implies that the line $u_1^0 x_1 + u_2^0 x_2 = u_0^0$ bisect each of the two sets into two halves. □

2. Other applications of fundamental groups

¶ Null-homotopic.

By definition any continuous map to a contractible space is null-homotopic.

Proposition 2.1. *We have*

(1) Any continuous map $f : S^2 \to S^1$ is null-homotopic.
(2) Any continuous map $f : \mathbb{RP}^2 \to S^1$ is null-homotopic.

Proof. (1) Since $\mathrm{Im}(f_*) = \{e\}$, f can be lifted to a map $f : S^2 \to \mathbb{R}$ which is null-homotopic. Thus $f = p \circ \tilde{f}$ is null-homotopic.
(2) Although $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2 \neq \{e\}$, the only group homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}$ is the trivial homomorphism. Thus we still have $\mathrm{Im}(f_*) = \{e\}$, and the same argument as above shows that f is null-homotopic. □

¶ Find all covering spaces.

By the classification theorem of covering spaces, the covering spaces of a (nice) topological spaces are in one-to-one correspondence with the subgroups of its fundamental group. Using this one may be able to find all covering spaces of a topological space. For example, in PSet11-2-3 you were asked to find all covering spaces of $S^1 \vee S^2$ and $S^1 \times S^1$. Here is another example: Since

$$\pi_1(\mathbb{RP}^n \times \mathbb{RP}^m) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{e, a\} \oplus \{e, b\}$$

with subgroups

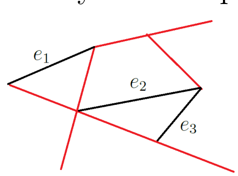
$$\{e\}, \{e, a\}, \{e, b\}, \{e, ab\}, \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

we conclude that the covering spaces of $\mathbb{RP}^n \times \mathbb{RP}^m$ are:

$$S^n \times S^m, \quad \mathbb{RP}^n \times S^m, \quad S^n \times \mathbb{RP}^m, \quad \mathbb{RP}^n \times \mathbb{RP}^m, \quad S^n \times S^m/(v, w) \sim (-v, -w).$$

¶ Use of van Kampen theorem: The fundamental groups of graphs.

Let's explain by an example.

$G =$ 

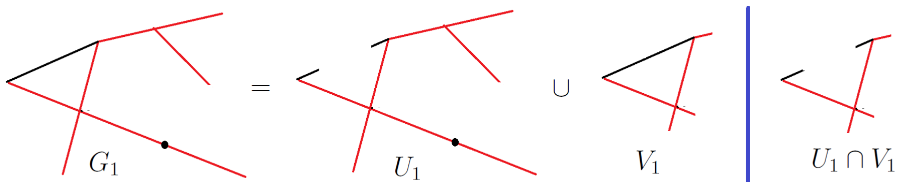
Let G_0 = a maximal tree in G (which is simply-connected and contains all vertices of G), as marked by red in the picture. Label the remaining edges by $e_1, e_2, e_3, \cdots$

To compute the fundamental group of G, we start with G_0. Since G_0 is a tree which is contractible,

$$\pi_1(G_0) \simeq \{e\}.$$

Now we add the edge e_1 and compute the fundamental group of $G_1 = G_0 \cup e_1$. The decomposition of G_1 into open subsets is shown as the following picture: Since $U_1 \cap V_1$ is contractible, while V_1 is homotopy equivalent to S^1, we get

$$\pi_1(G_1) \simeq \pi_1(G_0) *_{\{e\}} \mathbb{Z} = \mathbb{Z}.$$



Then we repeat the same procedure by adding e_2 to get

$$\pi_1(G_2) = \pi_1(G_1 \cup e_2) \simeq \pi_1(G_1) *_{\{e\}} \mathbb{Z} \simeq \mathbb{Z} * \mathbb{Z}$$

and finally

$$\pi_1(G) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

In general, given any connected graph G,

- there always exists a maximal tree G_0 (simply-connected and contains all vertices of G). [Proof: Induction for finite tree. Use Zorn lemma for general case.]
- If we let $\{e_\alpha\}$ be the set of all edges in G that are not in G_0, Then

$$\pi_1(G) \simeq *_{e_\alpha}\mathbb{Z}.$$

- For the case of finite graph G, if we denote

$$|V(G)| = \text{the number of vertices in } G$$

  and

$$|E(G)| = \text{the number of edges in } G,$$

  then the number of edges in G_0 is $|V(G)| - 1$. It follows

$$\pi_1(G) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k},$$

  where $k = |E(G)| - |V(G)| + 1$.

¶ Application to algebra: Nielsen-Schreier Theorem.

Not only algebra can be used to study topology, but also topology can be applied to give a quick proof of the following theorem in algebra:

Theorem 2.2 (Nielsen-Schreier)**.** *Any subgroup of a free group is still a free group.*

Sketch of proof. Let F be a free group, and H a subgroup. Let G be a linear graph s.t. $\pi_1(G) \simeq F$. One can check G is path-connected, locally path connected, and semi-locally simply connected. Thus there exists covering space $p : \widetilde{G} \to G$ s.t. $\pi_1(\widetilde{G}, \tilde{e}) \simeq H$. $\widetilde{G}$ is a linear graph and $\pi_1(\widetilde{G})$ is free. For details, see §83 and §84 of Munkres' book. □

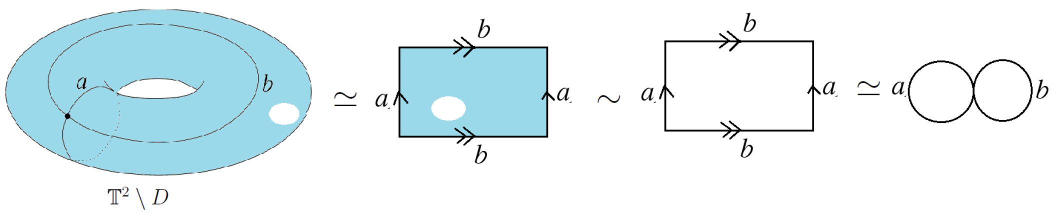
¶ Use of van Kampen theorem: The fundamental groups of $\mathbb{T}^2$ again.

Next we compute the fundamental group of $\Sigma_1 = \mathbb{T}^2$. Although we already know $\pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) \simeq \mathbb{Z}^2$, we want to re-compute its fundamental group via van Kampen, because the computation shed a light on how to compute the fundamental group of Σ_g (the closed oriented surface with g holes).

We first write $\mathbb{T}^2$ to a union of

$$U_1 = \mathbb{T}^2 \setminus \overline{D} \qquad \text{and } U_2 = \widetilde{D},$$

where D is a small disc and $\widetilde{D}$ is a small disc which contains $\overline{D}$. We first compute the fundamental group $\pi_1(\mathbb{T}^2 \setminus D)$. According to the following picture,



we have

$$\pi_1(U_1) \simeq \pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle.$$

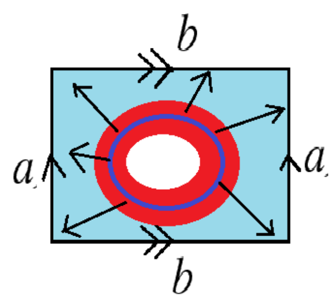
On the other hand, obvious U_2 is contractible, and $U_1 \cap U_2$ is a "thin annulus" which is homotopy equivalent to a circle, so

$$\pi_1(U_2) \simeq \pi_1(\text{"pt"}) = \{e\} \quad \text{ and } \quad \pi_1(U_1 \cap U_2) \simeq \pi(S^1) \simeq \mathbb{Z}.$$

Unfortunately the above information is still not enough to determine $\pi_1(\mathbb{T}^2)$, since we still need the map

$$\varphi = \iota_* : \pi_1(U_1 \cap U_2) \to \pi_1(U_1)$$

This group homotopy is induced by the inclusion map. According to the picture, the generator of $\pi_1(U_1 \cap U_2)$ is a circle which can be deformed inside U_1 to the boundary loop $aba^{-1}b^{-1}$. In other words,



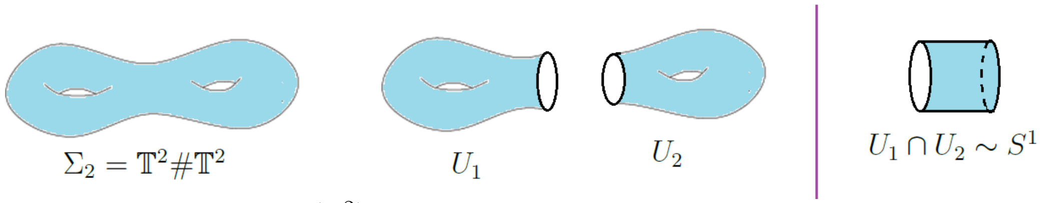
$$\varphi(1) = aba^{-1}b^{-1}$$

So we conclude

$$\pi_1(\mathbb{T}^2) \simeq (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} \{e\} = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \langle a, b \mid ab = ba \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

¶ Use of van Kampen theorem: The fundamental groups of $\Sigma_g = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$.

We start with $\Sigma_2 = \mathbb{T}^2 \# \mathbb{T}^2$, which can be decomposed as below:



Note that in computing $\pi_1(\mathbb{T}^2)$, we have already computed the fundamental group of U_1 and U_2. So we get

$$\pi_1(U_1) \simeq \langle a_1, b_1 \rangle, \quad \pi_1(U_2) \simeq \langle a_2, b_2 \rangle, \quad \pi_1(U_1 \cap U_2) \simeq \mathbb{Z}.$$

Moreover, we also explained just now that

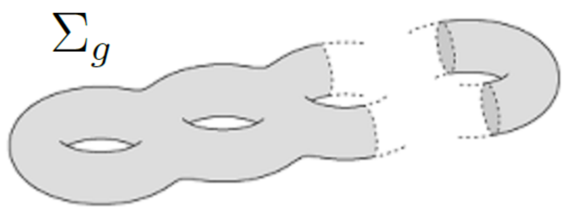
$$\varphi(1) = a_1 b_1 a_1^{-1} b_1^{-1}, \quad \psi(1) = b_2 a_2 b_2^{-1} a_2^{-1}.$$

It follows from van Kampen's theorem that

$$\pi_1(\mathbb{T}^2 \# \mathbb{T}^2) \simeq \langle a_1, b_1, a_2, b_2 \mid \underbrace{a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}}_{\varphi(1)\psi(1)^{-1}} = 1 \rangle$$

Note that this is no longer abelian.

In general by induction one can compute the fundamental group of $\Sigma_g = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$,



and the result is

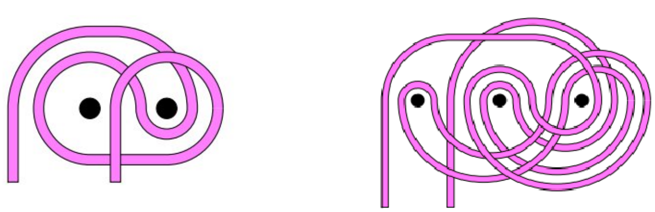
$$\pi_1(\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}) \simeq \langle a_1, b_1, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$$

¶ Application to recreational mathematics: Picture hanging.

Here is a picture hanging puzzle proposed by A. Spivak in 1997:

> How to hang a picture with several nails, such that none of the mails is removable, i.e. such that the picture will fall if any nail is removed?

Here are solutions for two or three nails:



In general the solution is not unique. So, what kind of twist is a solution? We may think of this problem via the fundamental group: For simplicity consider the two nail problem. Let $\gamma : [0,1] \to \mathbb{R}^2 \setminus \{p,q\}$ be a loop representing a solution. Then

$$[\gamma]_p \neq e \in \pi_1(\mathbb{R}^2 \setminus \{p,q\})$$

since the picture is hanged on the two nails without falling down. Since removing either nail will resulting a falling of picture, we get

$$[(i_p)_*\gamma]_p = e \in \pi_1(\mathbb{R}^2 \setminus \{q\}), \qquad [(i_q)_*\gamma]_p = e \in \pi_1(\mathbb{R}^2 \setminus \{p\}),$$

where $i_p : \mathbb{R}^2 \setminus \{p,q\} \hookrightarrow \mathbb{R}^2 \setminus \{p\}$ and $i_q : \mathbb{R}^2 \setminus \{p,q\} \hookrightarrow \mathbb{R}^2 \setminus \{q\}$ are the inclusion maps. If we denote $\pi_1(\mathbb{R}^2 \setminus \{q\}) \simeq \mathbb{Z} = \langle a \rangle, \pi_1(\mathbb{R}^2 \setminus \{p\}) \simeq \mathbb{Z} = \langle b \rangle$, then a solution to the two nail picture hanging puzzle is

$$[\gamma]_p = aba^{-1}b^{-1},$$

which is realized by the first picture. you can construct a loop $[\gamma]_p = aba^{-1}b^{-1}aba^{-1}b^{-1}$. You can also generalize this to more nails, e.g. for the four nail problem you may take the loop to be

$$aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}dcaba^{-1}b^{-1}c^{-1}bab^{-1}a^{-1}d^{-1} = [[[a,b],c],d]$$

or

$$aba^{-1}b^{-1}cdc^{-1}d^{-1}bab^{-1}a^{-1}dcd^{-1}c^{-1} = [[a,b],[c,d]].$$

For more discussions and generalizations of this puzzle, c.f.

https://arxiv.org/pdf/1203.3602.pdf

¶ Application to recreational mathematics: the plate trick.

Here is the description of the plate trick:

> Resting a small plate flat on the palm, it is possible to perform two rotations of one's hand while keeping the plate upright. After the first rotation of the hand, the arm will be twisted, but after the second rotation it will end in the original position.

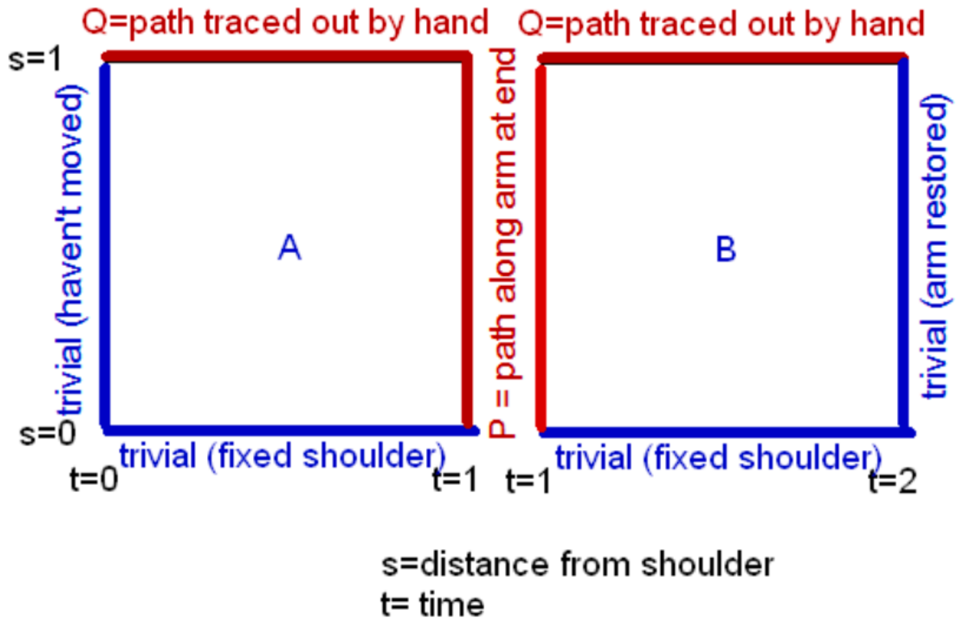
The mathematics behind the plate trick: $\pi_1(SO(3))$ has an order 2 element!

To explain, let's attach the standard orthogonal basis for $\mathbb{R}^3$ to each point along the arm. Now construct two different loops in $SO(3)$:

- the loop Q traced out by the fingertips over one rotation,
- the loop P traced out along the arm at the end of the first rotation.

We put them in the following picture, where the t parameter represents "time", so that at $t = 1$ you finished one rotation (and your arm is twisted), and at $t = 2$

you arm ends with the original position; and the s parameter represents your arm, so that $s = 0$ represents your shoulder and $s = 1$ represents your fingertips.



From the first square (from bottome+right to left+top in the square A in the picture) we see

$$[P]_p = [Q]_p.$$

From the two squares together (from top to bottom) we get

$$[Q]_p + [Q]_p = e.$$

Since at the end of one rotation, your arm is twisted and you can't untwist your arm while fixing your fingertips, we have

$$[P]_p \neq e.$$

So we conclude that the path homotopy class $[P]_p$ $(= [Q]_p)$ is an order two element in $\pi_1(SO(3))$!