Topology (H) Lecture 27
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Time: June 17, 2021

CLASSIFICATION OF CURVES

1. Classification of curves

Today we will study 1-manifolds, which are also known as "curves". The main theorem we want to prove the following classification theorem:

Theorem 1.1 (Classification of 1-manifolds)**.** *Up to homeomorphism, there are exactly 2 distinct connected* 1*-manifolds:* S^1 *and* $\mathbb{R}^1$*.*

Recall that if M is a 1-manifold, then for any $x \in M$, there exists an open neighborhood U and a homeomorphism $\varphi : U \to \mathbb{R}$.[1] The pair (φ, U) a *chart* near x. Although the proof of Theorem 1.1 is lengthy, the idea is clear: glue "neighbor charts" to larger charts until we can't glue any more.

¶ Intersection of two charts.

We first characterize the intersection of two charts:

Lemma 1.2. *Let* (φ_1, U_1) *and* (φ_2, U_2) *be two charts of a 1-manifold* M*, with* $U_1 \not\subset U_2$ *and* $U_2 \not\subset U_1$*. Suppose* $W \subset U_1 \cap U_2$ *is a connected component,* $\varphi_1(W) = (a, b)$ *and* $\varphi_2(W) = (c, d)$*, where* $a, b, c, d \in \mathbb{R} \cup \{\pm\infty\}$*. Moreover, suppose the "transition map"* $\varphi_{12} := \varphi_2 \circ \varphi_1^{-1} : (a, b) \to (c, d)$ *is monotonely increasing. Then we must have*

$$a \in \mathbb{R}, b = +\infty, c = -\infty, d \in \mathbb{R} \quad \text{or} \quad a = -\infty, b \in \mathbb{R}, c \in \mathbb{R}, d = +\infty.$$

Proof. We need to rule out all other cases. We know

$$a, c < +\infty \quad \text{and} \quad b, d > -\infty,$$

and from the condition "$U_1 \not\subset U_2$ and $U_2 \not\subset U_1$" we know

$$(a, b) \neq (-\infty, +\infty) \quad \text{and} \quad (c, d) \neq (-\infty, +\infty).$$

The remaining cases are

- $a, c \in \mathbb{R}$: So both $\varphi_1^{-1}(a)$ and $\varphi_2^{-1}(c)$ are well-defined points in M.
  First we claim that in this case, we must have $\varphi_1^{-1}(a) = \varphi_2^{-1}(c)$. Suppose on the contrary that $\varphi_1^{-1}(a) \neq \varphi_2^{-1}(c)$. Then for any small neighborhood U_a of $\varphi_1^{-1}(a)$, $\varphi_1(U_a)$ is an open neighborhood of a. Similarly for any

[1] Obviously in the definition of n-dimensional topological manifolds, we may replace the open set $V \subset \mathbb{R}^n$ by $\mathbb{R}^n$ itself.

small neighborhood U_c of $\varphi_2^{-1}(c)$, $\varphi_2(U_c)$ is an open neighborhood of c. Since $\varphi_{12} : (a,b) \to (c,d)$ is a homeomorphism which is increasing, it must map any small set $(a, a+\varepsilon_1)$ to a set $(c, c+\varepsilon_2)$. It follows that

$$\varphi_{12}(\varphi_1(U_a) \cap (a,b)) \cap \varphi_2(U_c) \neq \emptyset.$$

This implies

$$\varphi_2(U_a \cap \varphi_1^{-1}((a,b))) \cap \varphi_2(U_c) \neq \emptyset$$

and thus $U_a \cap U_c \neq \emptyset$. This contradicts with the fact M is (T2).

Now we have $\varphi_1^{-1}(a) = \varphi_2^{-1}(c)$. Then $\varphi_1^{-1}(a)$ is an interior point of U_2, and thus of $U_1 \cap U_2$. So a is an interior point of the set $\varphi_1(U_1 \cap U_2)$, which is a contradiction since a is a boundary point of the set $\varphi_1(W)$ which is a connected component of $\varphi_1(U_1 \cap U_2)$. So we can't have $a, c \in \mathbb{R}$.

- $b, d \in \mathbb{R}$: By the same argument as above we will get a contradiction. □

¶ Maps on intervals.

To study 1-manifolds, it is natural to study maps on intervals. We will need the following lemmas whose proofs are elementary and will be left as exercises:

Lemma 1.3. *Any continuous injective map $f : (a,b) \to \mathbb{R}$ must be strictly monotone.*

Lemma 1.4. *Let (φ_1, U_1) and (φ_2, U_2) be two charts and suppose*

$$\varphi_1(U_1 \cap U_2) = (a, +\infty) \quad \text{and} \quad \varphi_2(U_1 \cap U_2) = (-\infty, d),$$

where $a < d$ are real numbers. If

$$\varphi_{12} := \varphi_2 \circ \varphi_1^{-1} : (a, +\infty) \to (-\infty, d)$$

is increasing, then there exists a homeomorphism $\varphi : U = U_1 \cup U_2 \to \mathbb{R}$.

Lemma 1.5. *Let (φ_1, U_1) and (φ_2, U_2) be two charts and suppose $U_1 \cap U_2$ consists of two connected components W_1, W_2. If there exists $f < a < d < g$ with*

$$\varphi_1(W_1) = (a, +\infty), \qquad \varphi_2(W_1) = (-\infty, d),$$

$$\varphi_1(W_2) = (-\infty, f), \qquad \varphi_2(W_2) = (g, +\infty),$$

then there is a homeomorphism $\varphi : U_1 \cup U_2 \to S^1$.

¶ Gluing charts.

Now we can glue:

Proposition 1.6. *Suppose M is a 1-manifold. Let (φ_1, U_1) and (φ_2, U_2) are two charts. Then $U_1 \cap U_2$ has at most two connected components. Moreover,*

(1) If $U_1 \cap U_2$ is connected, then there exists a chart (φ, U) with $U = U_1 \cup U_2$.
(2) If $U_1 \cap U_2$ has two connected components, then $U_1 \cup U_2$ is homeomorphic to S^1.

Proof. We have three cases to discuss: $U_1 \cap U_2$ consists of one, two or at least three components. [It may happen that $U_1 \cap U_2 = \emptyset$, in which case there is nothing to prove.]

Case 1. $U_1 \cap U_2$ is connected.

If $U_1 \subset U_2$ or $U_2 \subset U_1$, the conclusion is trivial. So we assume $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$. Since $U_1 \cap U_2$ is connected and open, the images $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ must be connected open subsets of $\mathbb{R}$. So we have

$$\varphi_1(U_1 \cap U_2) = (a, b), \qquad \varphi_2(U_1 \cap U_2) = (c, d),$$

where $a, b, c, d \in \mathbb{R} \cup \{\pm\infty\}$. It follows that the "transition map"

$$\varphi_{12} := \varphi_2 \circ \varphi_1^{-1} : (a, b) \to (c, d)$$

is a homeomorphism. By Lemma 1.3, φ_{12} is monotone. Without loss of generality, we may assume φ_{12} is monotonely increasing (otherwise we may replace φ_1 by $-\varphi_1$).

By Lemma 1.2, we must have

$$a \in \mathbb{R}, b = +\infty, c = -\infty, d \in \mathbb{R} \quad \text{ or } \quad a = -\infty, b \in \mathbb{R}, c \in \mathbb{R}, d = +\infty.$$

So without loss of generality, we may assume

$$a \in \mathbb{R}, b = +\infty, c = -\infty, d \in \mathbb{R},$$

since for the other case we only need to swap (φ_1, U_1) and (φ_2, U_2). Again without loss of generality we may assume $a < d$ (otherwise we may add a constant to φ_2). Now the conclusion follows from Lemma 1.4.

Case 2. $U_1 \cap U_2$ has two connected components.

We denote the two connected components of $U_1 \cap U_2$ by W_1 and W_2. Following the same argument as in Case 1, we may simplify our situation to

$$\varphi_1(W_1) = (a, +\infty), \qquad \varphi_2(W_1) = (-\infty, d),$$

where $a < d$ are real numbers, and $\varphi_{12} = \varphi_2 \circ \varphi_1^{-1} : (a, +\infty) \to (-\infty, d)$ is monotonely increasing. On the other hand, by connectedness we have

$$\varphi_1(W_2) = (e, f), \quad \varphi_2(W_2) = (g, h).$$

Since $(a, +\infty) \cap (e, f) = \emptyset$, we must have $f \in \mathbb{R}$. Similarly $g \in \mathbb{R}$. This implies that $\varphi_{12} = \varphi_2 \circ \varphi_1^{-1} : (e, f) \to (g, h)$ is monotonely increasing, since otherwise we would have $(-\varphi_2) \circ \varphi_1^{-1} : (e, f) \to (-h, -g)$ is increasing, and $f, -g \in \mathbb{R}$, which contradicts with Lemma 1.2. So we can apply Lemma 1.2 to W_2 to get

$$\varphi_1(W_2) = (-\infty, f), \quad \varphi_2(W_2) = (g, +\infty).$$

Note that we have $f < a < d < g$. Now the conclusion follows from Lemma 1.5.

Case 3. $U_1 \cap U_2$ has at least three connected components.

This can never happen. In fact, suppose W_1, W_2 and W_3 are components of $U_1 \cap U_2$, then one of $\varphi(W_i)$ must be finite interval. By using Lemma 1.2 and the same arguments in Case 2 we see this can't happen. □

¶ Proof of the classification theorem.

Now we are ready to prove the classification theorem.

Proof of Theorem 1.1.

Since M is (A2), one can find a countable family of charts (φ_k, U_k) which cover M. Define $\widetilde{U}_1 = U_1$, and inductively define

$$\widetilde{U}_{n+1} = \widetilde{U}_n \cup U_{k(n)},$$

where $k(n)$ is the *smallest subscript k* such that $U_k \cap \widetilde{U}_n \neq \emptyset$ and $U_k \not\subset \widetilde{U}_n$.

Fact. We have $\bigcup_n \widetilde{U}_n = M$.

Proof. Denote $\widetilde{M} = \bigcup_n \widetilde{U}_n$. Then by definition, $\widetilde{M}$ is open and connected in M. Suppose $\widetilde{M} \neq M$. Take any $x \in M \setminus \widetilde{M}$. Take the smallest subscript m such that $x \in U_m$. We claim $U_m \cap \widetilde{M} = \emptyset$.

In fact, if there exists n such that $U_m \cap \widetilde{U}_n \neq \emptyset$, then $U_m \cap \widetilde{U}_{n'} \neq \emptyset$ for all $n' > n$. So there exists $l \leq m$ such that $U_{k(n+l)} = U_m$.

So $M \setminus \widetilde{M}$ is open. This contradicts with the connectedness of M. □

Now we apply Proposition 1.6. Take the *smallest* n, if exists, such that $\widetilde{U}_n \cap U_{k(n)}$ has two connected components. Then after gluing $\widetilde{U}_n$ is the domain of a chart, and thus $\widetilde{U}_{n+1}$ is homeomorphic to S^1, which is compact and thus closed in M since M is (T2). Since $\widetilde{U}_{n+1}$ is also open in M, we must have $M = \widetilde{U}_{n+1} \simeq S^1$.

Finally suppose at each step $\widetilde{U}_n \cap U_{k(n)}$ has only one connected components. Then we get an increasing sequence of open charts

$$\widetilde{U}_1 \subset \widetilde{U}_2 \subset \widetilde{U}_3 \subset \cdots$$

and chart maps $\tilde{\varphi}_n : \widetilde{U}_n \xrightarrow{\simeq} \mathbb{R}$. Without loss of generally, we may assume $\widetilde{U}_n \subsetneq \widetilde{U}_{n+1}$. If the sequence stops after finite steps, then we are done. If not, then we first choose a homeomorphism $\phi_1 : \widetilde{U}_1 \xrightarrow{\simeq} (0,1)$. After defining $\phi_n : \widetilde{U}_n \xrightarrow{\simeq} (a_n, b_n)$, we can inductively define $\phi_{n+1} : \widetilde{U}_{n+1} \xrightarrow{\simeq} (a_{n+1}, b_{n+1})$ such that

$$\text{either} \quad (a_{n+1}, b_{n+1}) = (a_n - 1, b_n) \quad \text{or} \quad (a_{n+1}, b_{n+1}) = (a_n, b_n + 1),$$

and such that $\phi_{n+1} = \phi_n \quad \text{on } \widetilde{U}_n$. Finally define $\varphi(x) = \phi_n(x)$ for $x \in \widetilde{U}_n$, we get a homeomorphism from M to $\mathbb{R}$ or $(m, +\infty)$ or $(-\infty, m)$, for the latter two cases we can compose with a homeomorphism to get a homeomorphism from M to $\mathbb{R}$. □

Remark 1.7. For topological 1-manifolds with boundary, we have

Theorem 1.8. *Up to homeomorphism, there are exactly 2 distinct connected 1-manifolds with boundary:* $[0,1]$ *and* $[0,1)$.

2. Knots and links: a quick glance

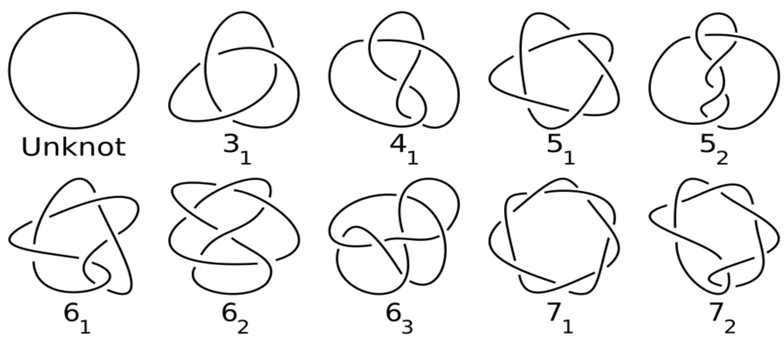
¶ Knots.

So up to homeomorphism, the circle S^1 is the only compact 1-manifold. It seems that up to this point, the study of compact 1-manifolds has draw to an end. But, NO! There are still much to say about 1-manifolds. Mathematics has a strong vitality!

In fact, there is a very active branch of mathematics, called *Knot theory*, in which people study how a circle is placed in the space $\mathbb{R}^3$.[2] Here we only give a very very brief introduction.

Definition 2.1. A *knot* K is the image of an embedding $f : S^1 \to \mathbb{R}^3$.

Here are some simple examples of knots:

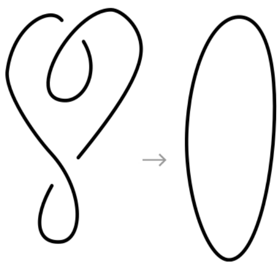


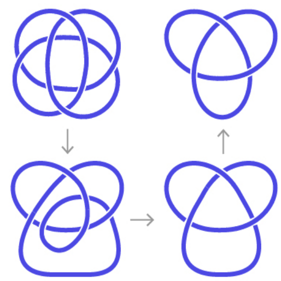
A knot is called a *polygonal knot* if it is a knot whose image in R^3 is the union of a finite set of line segments:

[2]We all know that in ancient China, people used knots to count and to record. A mathematical theory of knots was first developed in 1771 by A. Vandermonde (Yes, we have seen this name in linear algebra) who explicitly noted the importance of topological features when discussing the properties of knots related to the geometry of position. Mathematical studies of knots began with C. Gauss who defined the linking integral in a brief note on his diary in 1833 (which may or may not related to his study on Earth magnetism). In 1860s, Lord Kelvin proposed a theory that atoms were knots in the aether, and that different elements may be determined by the different possible knots. Although this physics theory was proved to be wrong, it led people to classify knots mathematically.

¶ Knot equivalence.

Like most mathematics theories, we would like to define equivalent knots and distinguish different knots. Roughly speaking, two knots are equivalent if you can "deform" one knot to another inside $\mathbb{R}^3$ without crossing.



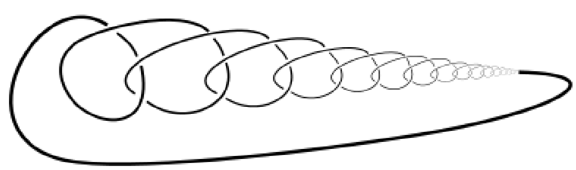


So in defining equivalent knots, we are not trying to find a homeomorphism between the two images K_1 and K_2 (both are homeomorphic to S^1), but trying to find a family of homeomorphisms of the ambient space $\mathbb{R}^3$, so that along with this family of ambient homeomorphisms, one knot is deformed to another.

Definition 2.2. Two knots K_1 and K_2 are *equivalent* if there exists a continuous mapping $H : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$, called an *ambient isotopy*, such that

(1) For each $t \in [0,1]$, the map $H(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism.
(2) $H(0,\cdot) = \mathrm{Id}_{\mathbb{R}^3}$.
(3) $H(1,K_1) = K_2$.

For simplicity, we only consider *tamed knots*, that is, those knots that are equivalent to *polygonal knots*. In particular, we will exclude "wild knots" like



¶ Knot group.

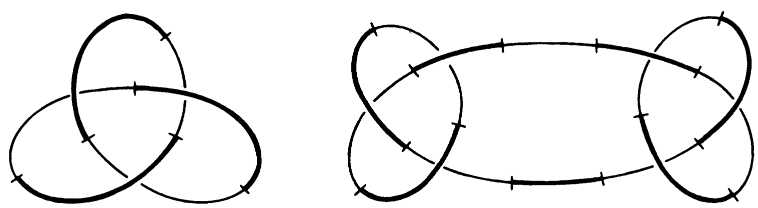
So how to distinguish different knots? Well, since the image of knots are all homeomorphic, it makes no sense to study $\pi_1(K)$. However, from Definition 2.2 we can easily see that equivalent knots have homeomorphic complements in $\mathbb{R}^3$. As a consequence, the fundamental group of the complements of equivalent knots are isomorphic. So it is natural to define

Definition 2.3. Let $K \subset \mathbb{R}^3$ be a knot. We call $\pi_1(\mathbb{R}^3 \setminus K)$ the *knot group* of K.

For example, we have seen in PSet 10-2-1 that the knot group of the unknot is

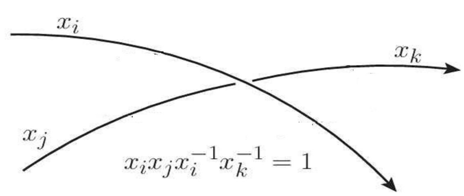
$$\pi_1(\mathbb{R}^3 \setminus \{(x, y, 0) \mid x^2 + y^2 = 1\}) \simeq \mathbb{Z}.$$

In general, there is a very simple way to write down the knot group of a tamed knot K: put the knot in "nice position" in the upper half space, so that the projection of K into the plane $z = 0$ is "nice". Break the knot into "overpasses" and "underpasses" according to this projection, as showed in the next picture:



Label all the "overpass" segments by $x_1, \cdots, x_m$. They are generators of the knot group.

Now look at the crossings. At each "crossing", there is one overpass segment, say x_i, and the "underpass segment" is divided into two pieces, which are connected to two overpass segments, say x_j and x_k. At such a crossing, there is an relation $x_i x_j = x_k x_i$.



Since "overpasses" and "underpasses" alternates, there are exactly n crossings. So there are n relations. With a little bit work, one can show that the last relation is redundant. So there are $n-1$ relations in total, which will be denoted by $r_1, \cdots, r_{n-1}$. Now we can write down a presentation of the knot group of a knot K:

Theorem 2.4. *The knot group of a (tamed) knot K is a finite presented group*

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_{n-1} \rangle.$$

where $x_1, \cdots, x_n, r_1, \cdots, r_{n-1}$ are described as above.

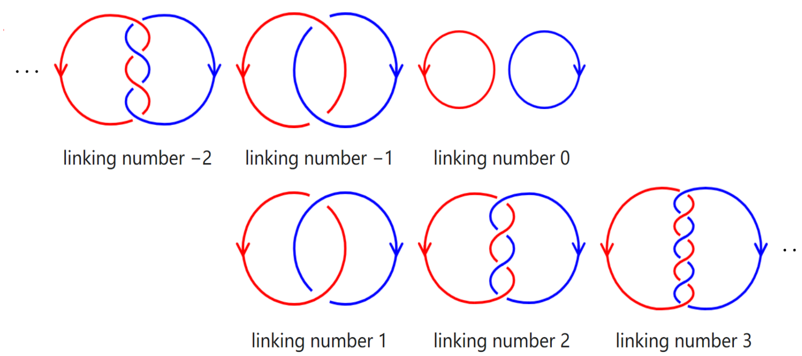
As one can easily guess, the proof is merely an application of van Kampen's theorem. For details of the proof, c.f. *M. Amstrong, Basic Topology*.

¶ Links.

Till now, when studying manifolds or other simple classes of topological spaces, we said something like "without loss of generality we may assume our object is connected, since disconnected objects are simply the union of their connected components (which is a finite union if our object is compact)". But it turns out that when studying compact curves in $\mathbb{R}^3$, disconnected curves makes our life much more complicated and much more interesting!

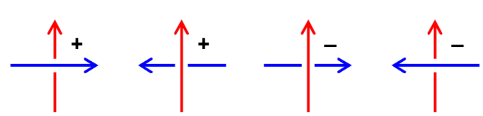
Definition 2.5. A *link* L is the image of an embedding of a disjoint union of finitely many circles in $\mathbb{R}^3$.

Here are some links with two circles:



As in the case of knots, one can define link equivalence and calculate the fundamental group $\pi_1(\mathbb{R}^3 \setminus L)$.

Given a link that consists of two circles (with natural orientations given by the embedding), one can define a numerical invariant called the *linking number*. Roughly speaking, if we put the two link in "nice" position, then there are four possible configuration of crossings. We mark each crossing by $+1$ or -1 according to the following picture:



Then the linking number is defined to be half of the summation of the marked numbers.

Amazingly, the linking number can be calculated via a double integral, first discovered by C. Gauss and thus is called *Gauss linking integral*:

$$\mathrm{Link}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \iint_{S^1 \times S^1} \frac{\det(\dot{\gamma}_1(s), \dot{\gamma}_2(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|^3} dsdt$$

where $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ are the embeddings of the two circles in $\mathbb{R}^3$.