Undergraduate Student's Topology(H) Course
An Introduction

Zuoqin Wang

February 21, 2022

Outline

1. What is topology...
   - A: The origin of the subject
   - B: Some abstract words
   - C: Some concrete pictures

2. Some topological theorems that we have learned/will learn
   - Part I: Point-Set Topology and Applications to Analysis
   - Part II: Algebraic/Geometric Topology

What's next ☞

A: The origin of the subject/name

Euler!

- The first topologist: Leonhard Euler (1707-1783)

Euler!

- The first topologist: Leonhard Euler (1707-1783)

He used the name

Analysis situs

Leibniz

- Euler credited the originating of the concept “analysis situs” to Gottfried Wilhelm Leibniz (1646-1716)

Leibniz

- Euler credited the originating of the concept "analysis situs" to Gottfried Wilhelm Leibniz (1646-1716)

(but what Leibniz had in his mind could be an entirely different thing ...)

Listing

- The first people used the word Topology=Topos+logos was Johann Benedict Listing (1808-1882): he created the word in a letter (1836) and published in a book(1847):

Listing

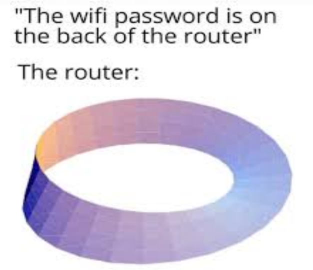
• The first people used the word Topology=Topos+logos was Johann Benedict Listing (1808-1882): he created the word in a letter (1836) and published in a book(1847):

He combined two Greek words

- topos($\tau o \pi o \varsigma$): place, location
- logos($\lambda o \gamma o \varsigma$): study

Mobius

- Listing was also the first one who discovered Möbius band (several months earlier than August Möbius (1790-1868))



Mobius

- Listing was also the first one who discovered Möbius band (several months earlier than August Möbius (1790-1868))

But Listing went further in exploring the properties of strips with higher-order twists instead of the non-orientability ...

Gauss!

- Listing was a student of Carl Friedrich Gauss (1777-1855)

Gauss!

- Listing was a student of Carl Friedrich Gauss (1777-1855)

Gauss also made great contributions to topology:

- the linking number/integral ⇝ knot theory, DNA ...
- Gauss-Bonnet-Chern theorem: curvature and topology

Riemann!

- Another student of Gauss: Bernhard Riemann (1826-1966)

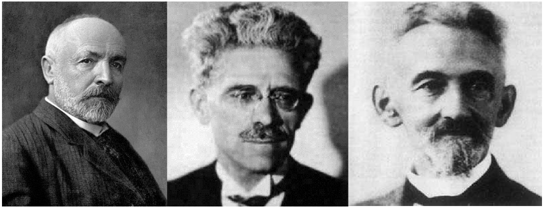
Riemann!

- Another student of Gauss: Bernhard Riemann (1826-1966)

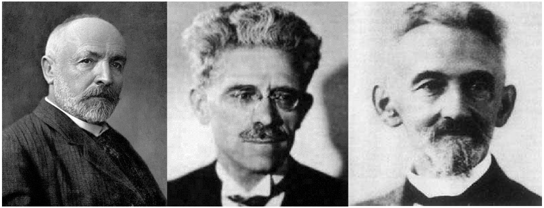
who also made great contributions to topology:

- Riemann surface ⇝ Topological methods in complex function theory
- Manifolds ⇝ the conception of abstract/higher-dim spaces
- Incorrect use of the Dirichlet Principle ⇝ compactness

Cantor, Frechet, and Hausdorff



Cantor, Frechet, and Hausdorff



In a totally different direction,

- Georg Cantor(1845-1918): abstract set theory $\rightsquigarrow$ limit points, ideas related to convergence of sequences(1870s)
- Maurice Frechet(1878-1973): axioms for convergence, and axioms for metric spaces (1905)
- Felix Hausdorff(1868-1942): axioms for topological spaces (Neighborhood approach)(1912)

Question: What is topology?

Question: What is topology?

Of course I can't give a simple answer to this question now. You should have your own answer at the end of this semester. Here let's see how mathematicians say:

It is a disease!

"Later generations will regard *Mengenlehre*[1] as a disease from which one has recovered."

Henry Poincare(1854-1912)[2]*, Congress of Mathematicians, Rome, 1908*

[1] "Mengenlehre" is the German word for *set theory* and *point set topology*
[2] Some people regard his paper *Analysis Situs*(1895) as the birth of topology

It is a disease!

"Later generations will regard *Mengenlehre*[1] as a disease from which one has recovered."

Henry Poincare(1854-1912)[2], *Congress of Mathematicians, Rome, 1908*

Contributions to topology:

- the foundation of topology for a space of any dimension
- basic concepts and invariants such as Betti numbers and the fundamental group.

[1] "Mengenlehre" is the German word for *set theory* and *point set topology*

[2] Some people regard his paper *Analysis Situs*(1895) as the birth of topology

It is an angel.

"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics."

Hermann Weyl, "Invariants", Duke Math. Journ., 1939

It is an angel.

"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics."

Hermann Weyl, "Invariants", Duke Math. Journ., 1939

Contributions to topology:

- utilized point-set topology to study Riemann surface ($\rightsquigarrow$ rigorous def. of manifolds)
- topological groups, Lie groups ($\rightsquigarrow$ symmetries in quantum mechanics)

Topology in China

Topology in China

中国拓扑学研究的开始

- 江泽涵(1902-1994): 中国拓扑学的奠基人，1939年开设“形势几何学”课程（西南联大）
- 陈省身(1911-2004): 整体微分几何之父，1946年讲授“拓扑学”（中研院数学所）
- 吴文俊(1919-2017): 著名拓扑学家，科大几何拓扑的源头

B: Some abstract words

Bourbaki: In mathematics we study structure

According to Bourbaki[3], in mathematics we study

1. structures defined on sets,
2. maps that preserve structures.

[3] Bourbaki is the pseudonym of a group of famous mathematicians, including Henri Cartan, Claude Chevalley, Jean Dieudonne, Jean-Pierre Serre, Alexandre Grothendieck, Andre Weil and many others. Bourbaki is most famous for its rigorous presentation of the series of books *Éléments de mathématique* and for introducing the notion of *structures* as the root of mathematics.

Bourbaki: In mathematics we study structure

According to Bourbaki[3], in mathematics we study

1. structures defined on sets,
2. maps that preserve structures.

Roughly speaking, a structure is a set endowed with some additional feathers on the set, usually prescribed via subsets (of subsets of ...).

[3] Bourbaki is the pseudonym of a group of famous mathematicians, including Henri Cartan, Claude Chevalley, Jean Dieudonne, Jean-Pierre Serre, Alexandre Grothendieck, Andre Weil and many others. Bourbaki is most famous for its rigorous presentation of the series of books *Éléments de mathématique* and for introducing the notion of *structures* as the root of mathematics.

Mother structures

There are many mathematical structures, among which the following three structures are most elementary (called *mother structures*):

- algebraic structure
- topological structure
- order structure

Mother structures

There are many mathematical structures, among which the following three structures are most elementary (called *mother structures*):

- algebraic structure
- topological structure
- order structure

(There are also many other structures: metric structure, measure structure, smooth structure, ...)

Topological structure

Roughly speaking, a topological structure, or a topology for short, defined on a set, is the structure using which one can talk about the conception of *neighborhoods* of an element.

Topological structure

Roughly speaking, a topological structure, or a topology for short, defined on a set, is the structure using which one can talk about the conception of *neighborhoods* of an element.

$\rightsquigarrow$ As a result, with topology at hand, one can talk about the conception of *continuity* for maps defined on such sets.

Topology v.s. analysis

- So topological structures lie in the center of analysis: it is the topology of $\mathbb{R}$ (or $\mathbb{R}^n$) that allows us to talk about the continuity of (multi-variable) functions.

[4]Warning: To generalize a theorem $\neq$ to write down the theorem in an abstract setting. Usually you will gain in two aspects: first the new theorem should have more applications, and second the abstract form should be able to help you to understand the nature of the original theorem itself.

Topology v.s. analysis

• So topological structures lie in the center of analysis: it is the topology of $\mathbb{R}$ (or $\mathbb{R}^n$) that allows us to talk about the continuity of (multi-variable) functions.

• Many conceptions and theorems we learned in mathematical analysis are topological, and one of the main goals of the first half of this course (which is usually called general topology or point-set topology) is trying to extend these conceptions and theorems to more general spaces. [4]

[4]Warning: To generalize a theorem $\neq$ to write down the theorem in an abstract setting. Usually you will gain in two aspects: first the new theorem should have more applications, and second the abstract form should be able to help you to understand the nature of the original theorem itself.

Topology v.s. geometry

• Topological structures are also most important in geometry: they tell us how points cluster together in a space, without introducing further structures like distance. In particular, you can change the shape by stretching or twisting or bending. Sometimes topology is called "rubber-sheet geometry"[5]

[5]We will not use such name because it is in fact very misleading.

Topology v.s. geometry

• Topological structures are also most important in geometry: they tell us how points cluster together in a space, without introducing further structures like distance. In particular, you can change the shape by stretching or twisting or bending. Sometimes topology is called "rubber-sheet geometry"[5]

• In the second half of this course we will focus on geometric properties that are induced by topology (and not metric).

[5] We will not use such name because it is in fact very misleading.

C: Some concrete pictures

Topology as a branch of mathematics

- As a branch of mathematics, topology is the study of qualitative and quantitative properties of certain objects (called topological spaces) that are invariant under a certain kind of transformations (continuous maps).

Topology as a branch of mathematics

• As a branch of mathematics, topology is the study of qualitative and quantitative properties of certain objects (called topological spaces) that are invariant under a certain kind of transformations (continuous maps).

Let's explain this sentence by some pictures.

Coffee mug v.s. doughnut

We start with an old joke:

"A topologist is someone who can't tell the different between a coffee cup and a doughnut."

Computational mistakes...

Topologists always make mistakes when doing computations because ...

Computational mistakes...

Topologists always make mistakes when doing computations because ...

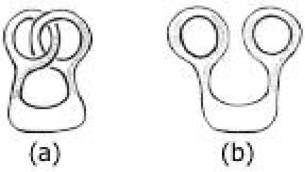
$$1=2 \qquad \text{ONE}=\text{TWO}$$

$$3\neq 4 \qquad \text{THREE}\neq\text{FOUR}$$

in topology!

Unlock two rings

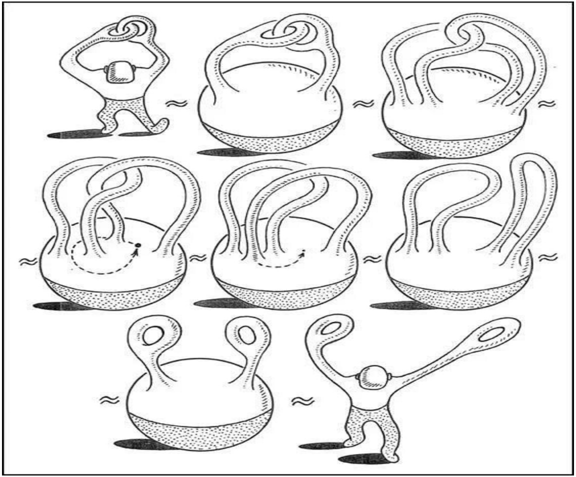
Here is another interesting problem:



Two rings are locked together. Can you unlock the two rings without breaking any of them?

Unlock two rings: The solution

Here is the solution:



Königsberg seven bridge problem

The city of Königsberg[6] consists of four lands. They are connected by seven bridges. The problem is:

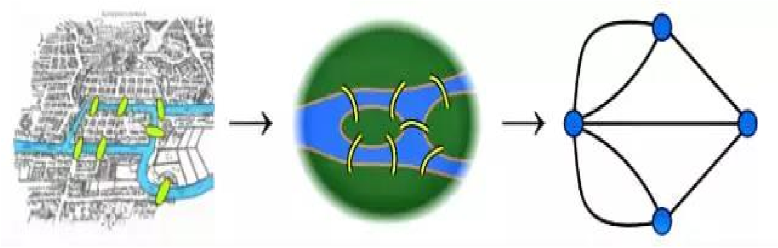
> *Is it possible for a pedestrian to walk across all seven bridges without crossing any bridge twice?*

[6] It is the hometown of the great philosopher I. Kant and the great mathematician D. Hilbert. It was renamed Kaliningrad in 1945.

Königsberg seven bridge problem

The city of Königsberg[6] consists of four lands. They are connected by seven bridges. The problem is:

> *Is it possible for a pedestrian to walk across all seven bridges without crossing any bridge twice?*



[6] It is the hometown of the great philosopher I. Kant and the great mathematician D. Hilbert. It was renamed Kaliningrad in 1945.

Euler's solution to Königsberg seven bridge problem

• The problem did not fit in any existing mathematical framework at that time. It looks like a problem in geometry, but there was no need for distances. Only the relative positions are needed.

[7] By the way, in 1875 the city built a new bridge and the answer became yes!

Euler's solution to Königsberg seven bridge problem

• The problem did not fit in any existing mathematical framework at that time. It looks like a problem in geometry, but there was no need for distances. Only the relative positions are needed.

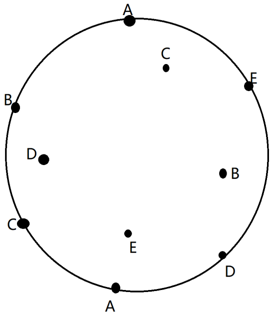
• The birth of topology (and graph theory): Euler solved this problem in 1736 and the answer is no[7]:

- Euler wrote to his friend Ehler in 1736, "*this type of solution bears little relationship to mathematics....*"
- Later in 1736 he wrote to Marinoni, "*it occurred to me to wonder whether it belonged to the geometry of position, which Leibniz had once so much longed for*".
- Finally, still in 1736, he solved this problem and generalize his solution to general case.

[7] By the way, in 1875 the city built a new bridge and the answer became yes!

Connecting points

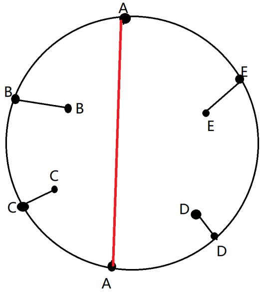
Here is another example that you can apply the idea of topology in solving problems:



Question: Can you connect points marking the same letter by non-intersecting curves inside the circle?

Connecting points: Solution

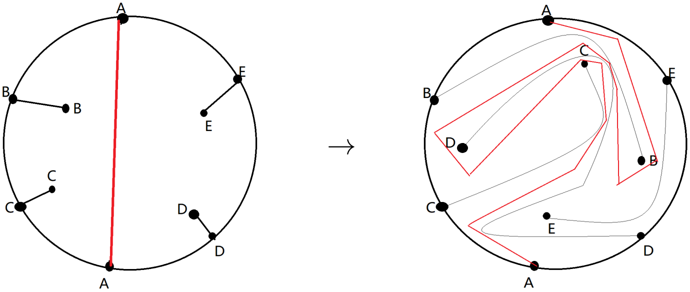
To solve this problem "topologically", you first continuously move the points inside the circle to positions that you can easily solve the problem. Then continuously move the points back and stretch the curves you have drawn so that they are always non-intersecting:



$\longrightarrow$

Connecting points: Solution

To solve this problem "topologically", you first continuously move the points inside the circle to positions that you can easily solve the problem. Then continuously move the points back and stretch the curves you have drawn so that they are always non-intersecting:



What's next ☞

1. What is topology...
   - A: The origin of the subject
   - B: Some abstract words
   - C: Some concrete pictures
2. Some topological theorems that we have learned/will learn
   - Part I: Point-Set Topology and Applications to Analysis
   - Part II: Algebraic/Geometric Topology

As we mentioned above, topology structure is one of the mother structures in mathematics. In fact, we have used topology in many places in earlier courses. Here we list some of them, together with some related theorems that we might cover later in this course.

Part I: Point-Set Topology and Applications to Analysis

A: The Intermediate Value Theorem

We start with a baby theorem:

Baby-Theorem

Suppose you are given a point A in the upper half plane, and a point B in the lower half plane. If you draw a planar curve connecting A to B, the curve must intersect the x-axis.

A: The Intermediate Value Theorem

• At the very beginning of mathematical analysis, we learned following more general theorem, which is one of the most important theorems in mathematics:

Theorem (The Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any value y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ s.t. $f(c) = y$.

A: The Intermediate Value Theorem

• At the very beginning of mathematical analysis, we learned following more general theorem, which is one of the most important theorems in mathematics:

Theorem (The Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any value y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ s.t. $f(c) = y$.

• Topological property behind the Intermediate Value Theorem:

the *connectedness* of $[a, b]$.

A: The Intermediate Value Theorem

In this course we will prove:

Theorem (The Generalized Intermediate Value Theorem)

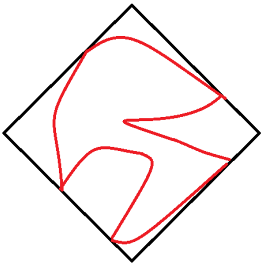
Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous map. Then for any connected subset $A \subset X$, the image $f(A)$ is a connected subset in Y.

A: The Intermediate Value Theorem

Here is a simple but beautiful application of IVT:

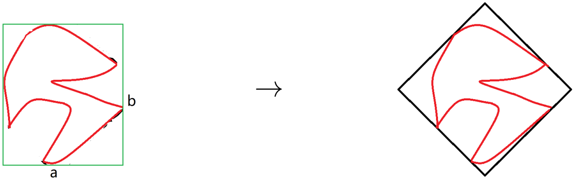
Proposition

Let $\mathcal{C}$ be a closed planar curve. Then there exists a planar square that contains the curve $\mathcal{C}$, so that the curve touches all four sides (vertices count as points in both sides) of the square.



A: The Intermediate Value Theorem

The proof: To see this you first draw a rectangle satisfying the given property. Then you rotate the rectangle continuously, to get new rectangles (with different side lengths a and b satisfying the same requirement). As you rotate, the value of $a - b$ changes continuously. Moreover, it will change sign after rotating 90 degrees. So it must attain the value 0 during the rotation.



The square peg problem

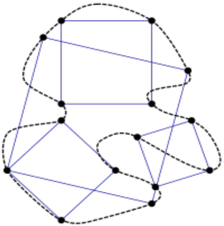
Before we move to next topic, we mention a similar problem in planar geometry. It is proposed by O. Toeplitz in 1911, and is known as the *inscribed square problem*, or the *square peg problem*. It is still unsolved today!

Conjecture (Toeplitz)

On every continuous simple closed planar curve, one can find four points that are the vertices of a square.

The square peg problem

Here is an example on which you can find four such squares:



The square peg problem

Some developments on the square peg problem:

1. Emch (1916): true for smooth convex curves, piecewise analytic curves (by using some kind of IVT argument)
2. Schnirelman 1929: true for smooth curves

The square peg problem

Some developments on the square peg problem:

1. Emch (1916): true for smooth convex curves, piecewise analytic curves (by using some kind of IVT argument)
2. Schnirelman 1929: true for smooth curves
3. Vaughan 1977: On every continuous simple closed planar curve, one can find four points that are the vertices of a rectangle.
    1. Idea: use the topology of the Möbius strip!

The square peg problem

Some developments on the square peg problem:

1. Emch (1916): true for smooth convex curves, piecewise analytic curves (by using some kind of IVT argument)
2. Schnirelman 1929: true for smooth curves
3. Vaughan 1977: On every continuous simple closed planar curve, one can find four points that are the vertices of a rectangle.
    1. Idea: use the topology of the Möbius strip!
4. Greene-Lobb 2020: For any rectangle, on any smooth simple closed planar curve, one can find four points that are the vertices of a rectangle.
    1. Idea: use symplectic topology!

B: The Extremal Value Theorem

For the second topic, again we start with a baby theorem:

Baby-Theorem

Every finite subset of $\mathbb{R}$ contains its supremum/infimum.

B: The Extremal Value Theorem

In mathematical analysis, we learned

Theorem (The Extremal Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains both a maximal value and a minimal value.

B: The Extremal Value Theorem

In mathematical analysis, we learned

Theorem (The Extremal Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains both a maximal value and a minimal value.

- Easy to see
  - The theorem fails if you replace $[a, b]$ by (a, b) or $[a, +\infty)$.
  - The theorem holds if you replace $[a, b]$ by $[a, b] \cup [c, d]$.

B: The Extremal Value Theorem

- Topological property behind the Extremal Value Theorem:

$$\boxed{\textit{compactness} \text{ of } [a, b].}$$

(We will see: compactness is a generalization of finiteness.)

B: The Extremal Value Theorem

- Topological property behind the Extremal Value Theorem:

compactness of $[a, b]$.

(We will see: compactness is a generalization of finiteness.)

- We can easily extend the EVT to

Theorem (The Generalized Extremal Value Theorem)

Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous map. Then for any compact subset $A \subset X$, the image $f(A)$ is a compact subset in Y.

Compactness

Compactness is one of the most important conception in topology. It is widely used in many other branches of mathematics, especially in analysis.

Dirichlet't principle

Here is some history: In 1856-1857, when lectured on potential theory, Peter Dirichlet (1805-1859) proposed a method, known as *Dirichlet't principle*[8], to solve the equation

$$\Delta u(x, y, z) = 0$$

on a region $\Omega \subset \mathbb{R}^3$ with prescribed boundary value f on the boundary surface $S = \partial\Omega$.

[8] named by Riemann, but the method was used earlier by Green, Gauss, ...

Dirichlet's principle

- To solve this he considered the integral

$$U = \int_\Omega \left[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 \right] dV.$$

which is always non-negative. Dirichlet concluded that there must be at least one function u on Ω for which the integral reaches a minimum. Then by using a standard variational argument one can show that the minimizer satisfies $\Delta u = 0$.

Dirichlet's principle

- To solve this he considered the integral

$$U = \int_\Omega \left[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 \right] dV.$$

which is always non-negative. Dirichlet concluded that there must be at least one function u on Ω for which the integral reaches a minimum. Then by using a standard variational argument one can show that the minimizer satisfies $\Delta u = 0$.

- Wait! As criticised by Weierstrass, there was a problem: The existence of a greater lower bound for the values of the integral does not necessarily imply the existence of a minimizer. To guarantee this, one need compactness (where we view functions as points).

Compactness: Arzela-Ascoli

The classical Arzela-Ascoli theorem is one of the first attempts trying to solve this type of problems. They justified Dirichlet's principle under extra assumptions:

Theorem (Arzela-Ascoli)

Let $\{f_n\}$ be a uniformly bounded and equicontinuous sequence of real-valued continuous functions defined on $[a, b]$. Then there exists a subsequence that converges uniformly.

Compactness: Arzela-Ascoli

In this course we will prove a much general version:

Theorem (Generalized Arzela-Ascoli Theorem)

Let $\mathcal{F}$ be a point-wise bounded and equicontinous family of functions defined on a compact space X. Then for any sequence in $\mathcal{F}$, there exists a subsequence that converges uniformly.

Compactness: The Blaschke selection theorem

• Arzela-Ascoli theorem is widely used in functional analysis and partial differential equation to prove the existence of a limit. Here we mention an equivalent form of the Arzela-Ascoli theorem in convex geometry:

Theorem (The Blaschke selection theorem)

Given a sequence $\{K_n\}$ of convex sets contained in a bounded set, there exists a subsequence $\{K_{n_m}\}$ and a convex set K such that K_{n_m} converges (in a sense that will be clear later) to K.

Compactness: The Blaschke selection theorem

• The Blaschke selection theorem can be used to prove the existence of a solution to many geometric problems (but the solution is still unknown). E.g.:

- Lebesgue's universal covering problem: the convex shape of smallest area that can cover any planar set of diameter one. (Note that a circle with diameter 1 is NOT a solution because it can't cover the Reuleaux triangle.) (Known: it should has area between 0.832 and 0.8440935944)

C: Separation properties

- Again we start with a baby theorem:

Baby-Theorem

Suppose $a < b < c < d$. Then one can find a continuous function $f : \mathbb{R} \to \mathbb{R}$ s.t. $f([a, b]) = 0, f([c, d]) = 1$.

C: Separation properties

- Again we start with a baby theorem:

Baby-Theorem

Suppose $a < b < c < d$. Then one can find a continuous function $f : \mathbb{R} \to \mathbb{R}$ s.t. $f([a,b]) = 0, f([c,d]) = 1$.

- It is not hard to extend this baby theorem to

Theorem (Urysohn Lemma in $\mathbb{R}^n$)

Suppose $K, L \subset \mathbb{R}^n$ are closed sets, and $K \cap L = \emptyset$. Then $\exists$ continuous function $f : \mathbb{R}^n \to \mathbb{R}$ s.t. $f(K) = 0, f(L) = 1$.

C: Separation properties

- Urysohn's lemma is useful because it is equivalent to the following extension theorem:

Theorem (Tietze Extension Theorem)

Suppose $K \subset \mathbb{R}^n$ is closed, f is a continuous function defined on K. Then f can be extended to a continuous function on $\mathbb{R}^n$.

C: Separation properties

• Urysohn's lemma is useful because it is equivalent to the following extension theorem:

Theorem (Tietze Extension Theorem)

Suppose $K \subset \mathbb{R}^n$ is closed, f is a continuous function defined on K. Then f can be extended to a continuous function on $\mathbb{R}^n$.

• Topological property behind these two Theorems:

separation properties of $\mathbb{R}^n$.

C: Separation properties

Again we can extend these two theorems to more general topological spaces. For example, we will prove

Theorem (Tietze extension theorem)

Suppose X is a normal topological space, and $A \subset X$ is closed. Then any continuous function $f : A \to \mathbb{R}$ can be extended to a continuous function $F : X \to \mathbb{R}$.

C: Separation properties

Both compactness and the separation property will be needed in proving the famous Stone-Weierstrass theorem (in a general setting):

Theorem (Stone-Weierstrass)

Let X be any compact Hausdorff space, and $L \subset C(X, \mathbb{R})$ be a sub-algebra which vanishes at no point and separate points. Then L is dense.

Part II: Algebraic/Geometric Topology

D: Brouwer Fixed Point Theorem

Again we start with a baby theorem that we learned in mathematical analysis:

Baby-Theorem (Baby fixed point theorem)

Let $f : [a, b] \to [a, b]$ *be any continuous function. Then* $\exists c \in [a, b]$ *s.t.* $f(c) = c$.

D: Brouwer Fixed Point Theorem

In this course we will prove

Theorem (Brouwer Fixed Point Theorem for $n = 2$)

Let $\mathbb{D}$ be a planar disc and $f : \mathbb{D} \to \mathbb{D}$ be any continuous map. Then $\exists p \in \mathbb{D}$ s.t. $f(p) = p$.

D: Brouwer Fixed Point Theorem

Some thoughts:

- Obviously the fixed point theorem fails if you replace $[a, b]$ by $(a, b]$ or $[a, b] \cup [c, d]$, which indicates the need of compactness and connectedness.
- It also fails if you replace $\mathbb{D}$ by a closed planar annulus. The reason is that you can find a circle in the annulus which cannot be deformed to a point. This is a *higher level connectedness*, which is described by the theory of *fundamental groups* (and more generally, *homotopy groups*).

D: Borsuk-Ulam theorem

After developing the theory of fundamental groups, we will prove the 2-dimensional Brouwer fixed point theorem as well as the Borsuk-Ulam theorem for $n = 2$:

Theorem (Borsuk-Ulam Theorem)

There does not exists any antipodal continuous map $f : S^n \to S^{n-1}$.

D: The Ham-Sandwich Theorem

As a consequence of Borsuk-Ulam theorem ($n = 3$), one has

Theorem (The Ham-Sandwich Theorem)

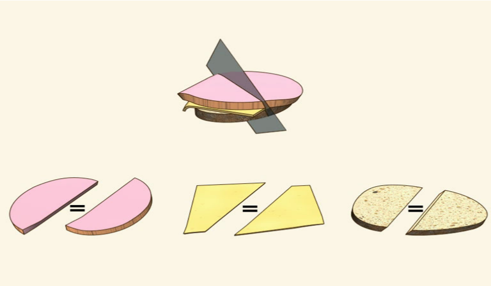
Given any sandwich consisting of bread, cheese and ham, one can cut the sandwich into two pieces by a single knife slice so that each piece has exactly half bread, half cheese and half ham.

D: The Ham-Sandwich Theorem

As a consequence of Borsuk-Ulam theorem ($n = 3$), one has

Theorem (The Ham-Sandwich Theorem)

Given any sandwich consisting of bread, cheese and ham, one can cut the sandwich into two pieces by a single knife slice so that each piece has exactly half bread, half cheese and half ham.



D: Brouwer Fixed Point Theorem

Again these theorems generalize to higher dimension. However, the proofs are more involved: the fundamental groups are not enough. We will not try to develop the general theory of homotopy groups or homology groups to prove these theorems. However, we will prove the general Brouwer fixed point theorem via ideas from *differential topology*:

Theorem (Brouwer Fixed Point Theorem for arbitrary n)

Let $\mathbb{B} \subset \mathbb{R}^n$ be a ball and $f : \mathbb{B} \to \mathbb{B}$ be any continuous map. Then $\exists p \in \mathbb{B}$ s.t. $f(p) = p$.

D: Brouwer Invariance of domain

With some hard work, we can prove

Theorem (Brouwer's Invariance of domain)

If $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^n$ is injective and continuous, then $f(U)$ is open in $\mathbb{R}^n$.

D: Brouwer Invariance of domain

With some hard work, we can prove

Theorem (Brouwer's Invariance of domain)

If $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^n$ is injective and continuous, then $f(U)$ is open in $\mathbb{R}^n$.

which implies the following theorem that looks "obviously true":

Theorem (Topological of dimension)

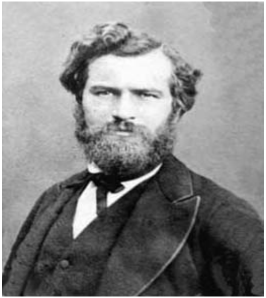
If $n \neq m$, then (in topology) $\mathbb{R}^n \neq \mathbb{R}^m$.

E: Jordan curve theorem

Here is another theorem which "is obviously true" but the proof is very complicated:

Theorem (Jordan Curve Theorem)

Any simple closed curve in the plane separate the plane into two disjoint regions.

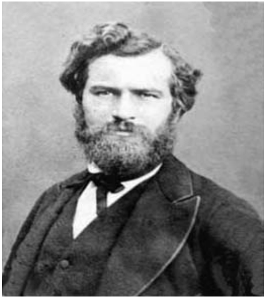


E: Jordan curve theorem

Here is another theorem which "is obviously true" but the proof is very complicated:

Theorem (Jordan Curve Theorem)

Any simple closed curve in the plane separate the plane into two disjoint regions.

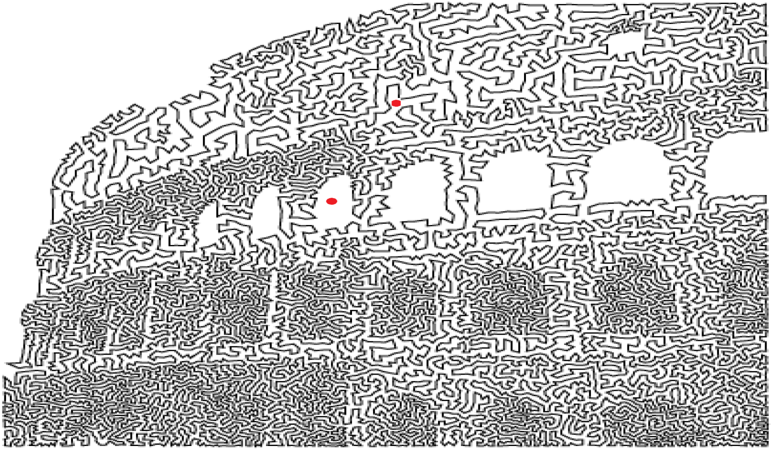


Camille Jordan (1838-1922) was also known for:

- Jordan normal form (linear algebra)
- Jordan measure (real analysis)

E: Jordan curve theorem

A simple closed curve need not be very simple:



E: Jordan-Brouwer

Similar (but weaker) theorem holds in higher dimension, which we will not prove in this course:

Theorem (Jordan-Brouwer separation theorem)

If X is a subset in $\mathbb{R}^n$ which is homeomorphic to S^{n-1}, then $\mathbb{R}^n \setminus X$ has exactly two components, one is bounded and the other is unbounded, so that X is their common boundary.

F: Euler Polyhedron Formula

Finally let's return to Euler's work. In 1750 Euler proved one of his most famous formula[9]:

Theorem (Euler Polyhedron Formula, 1750)

For any convex polyhedron,

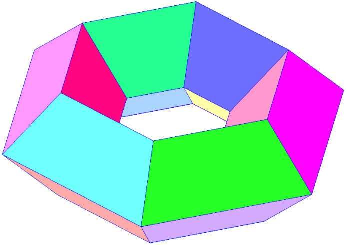
$$V - E + F = 2,$$

where V, E, F denotes the number of vertices, edges and faces respectively.

[9] Some people regard this theorem as the birth of topology.

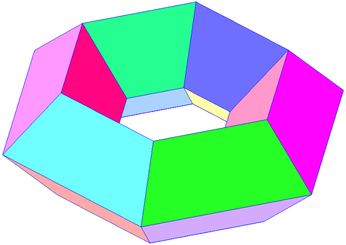
F: Euler Polyhedron Formula

You may ask: what about other polyhedrons? For example,



F: Euler Polyhedron Formula

You may ask: what about other polyhedrons? For example,



If you count the number of vertices, edges and faces of this torus-like polyhedron, you will find

$$V - E + F = 24 - 48 + 24 = 0.$$

F: Euler Polyhedron Formula

It turns out that the numbers 2 and 0 are topological:

- 2 is the Euler characteristic of S^2 (which is homeomorphic to convex polyhedrons in $\mathbb{R}^3$)
- 0 is the Euler characteristic of the torus T^2.

F: Euler Polyhedron Formula

In general, one has

Theorem (Generalized Euler Polyhedron Formula, L'Huilier 1812)

If polyhedron $P \simeq$ surface S, then

$$V - E + F = \chi(S).$$

F: Euler Polyhedron Formula

In general, one has

Theorem (Generalized Euler Polyhedron Formula, L'Huilier 1812)

If polyhedron $P \simeq$ surface S, then

$$V - E + F = \chi(S).$$

Here, the Euler characteristic $\chi(S)$ of an (oriented) closed surface S is defined to be

$$\chi(S) = 2 - 2k,$$

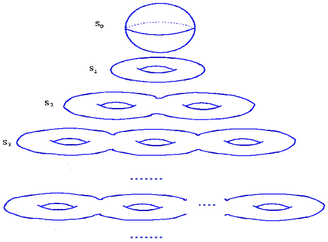
where k equals "the number of holes" of S.

F: Classification of surfaces

We will prove the following classification theorem of surfaces[10]:

Theorem (Classification theorem of oriented compact surfaces)

The list of all oriented compact surfaces without boundary:



[10]We will prove a more general version: the classification of compact surfaces (including non-orientable surfaces and compact surfaces with boundary).

F: Poincaré-Hopf Theorem in 2-dim

As an application, we can prove

Theorem (Poincaré-Hopf)

Let S be a connected compact oriented surface without boundary, and V a continuous vector field on S with only isolated critical points. Then

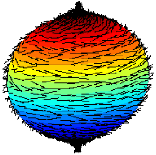
$$\sum_p \mathrm{ind}_p V = \chi(S).$$

F: Hairy ball theorem

Since $\chi(S^2) = 2 \neq 0$, we get

Corollary (Hairy ball theorem)

Any continuous vector field on the sphere S^2 must vanish at some point.



Two sayings

Let's end this lengthy introduction by two sayings:

Mathematicians do not study objects, but relations between objects.

– H. Poincare

"Obvious" is the most dangerous word in mathematics.

– E. Bell