PROBLEM SET 1, PART 1: TOPOLOGY (H)
DUE: FEBRUARY 28, 2022, BEFORE CLASS

(1) [Topology of symbols]
Classify the following symbols according to the topology of their pictures:

$$\Pi, \Sigma, \Psi, \Phi, \Gamma, \Upsilon, \Omega, \Theta, \Xi, \heartsuit$$

$$\phi, \varphi, \pi, \theta, \alpha, \beta, \gamma, \mu, \tau, \delta, \epsilon$$

$$+, \times, \otimes, \nabla, \cup, \sim, \propto, \rightarrow$$

(2) [Fake soccer!]
In Figure 1 you can see a soccer ball that I found from the internet. Obviously the careless designer never learned topology. Explain why.

FIGURE 1. Fake soccer ball

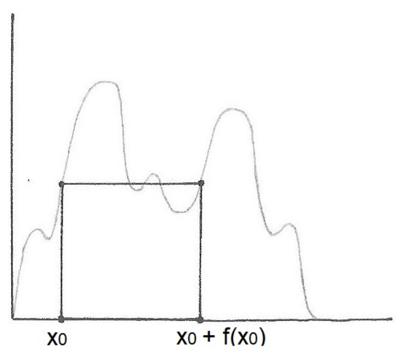

FIGURE 2. Inscribed square

(3) [Inscribed square problem: a simple case]
Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1) = 0$. Consider the simple closed curve C that consists of the graph of f and the line segment of the x-axis from $x = 0$ to $x = 1$. Prove: One can find four points on C that are the vertices of a square. [Hint: Consider the function $g(x) = f(x) - f(x + f(x))$. See Figure 2.]

(4) [Weierstrass's counterexample to Dirichlet principle]
For any $u \in \mathcal{A} = \{C^1([-1,1]) \mid u(-1) = 0, u(1) = 1\}$, define

$$F(u) = \int_{-1}^{1} |xu'(x)|^2 dx.$$

(a) Prove: For each $n \in \mathbb{N}$, the function

$$u_n(x) := \left(\sin \frac{n\pi x}{2}\right)^2 \chi_{[0,1/n]}(x) + \chi_{(1/n,1](x)}$$

is an element in $\mathcal{A}$ (where $\chi_A(x)$ is the characteristic function of the set A).

(b) Prove: $\lim_{n\to\infty} F(u_n) = 0$.

(c) Prove: There is no function $u \in \mathcal{A}$ that attains the minimum of F.