TOPOLOGY (H), PROBLEM SET 1, PART 2
DUE: FEBRUARY 28, 2022, BEFORE CLASS

(1) [Pseudo-metric]
A *pseudo-metric* on a set X is a map $d : X \times X \to [0, +\infty)$ that satisfies
- $d(x, x) = 0$. [Note: this is weaker than being a metric.]
- $d(x, y) = d(y, x)$.
- $d(x, y) + d(y, z) \geq d(x, z)$.

Let (X, d) be a pseudo-metric space. Define an *equivalence relation* on X via
$$x \sim y \Longleftrightarrow d(x, y) = 0.$$
Let $\overline{X} = X/\sim$ be the quotient (i.e. the set of equivalent classes), and let $p : X \to \overline{X}$ be the quotient map. Prove: there is a unique metric $\bar{d}$ on $\overline{X}$ so that
$$d(x, y) = \bar{d}(p(x), p(y)).$$

(2) [Metric-preserving functions]
Let $f : [0, +\infty) \to [0, +\infty)$ be a function (which need not be continuous). We say f is a *metric-preserving function* if for any metric space (X, d), the map $\widetilde{d} : X \times X \to \mathbb{R}$ defined by $\widetilde{d}(x, y) := f(d(x, y))$ is a metric on X.
(a) Prove: If f is a metric-preserving function, then $f^{-1}(\{0\}) = \{0\}$ and f is sub-additive:
$$f(\alpha + \beta) \leq f(\alpha) + f(\beta), \qquad \forall \alpha, \beta \in [0, +\infty).$$
(b) Prove: a function $f : [0, +\infty) \to [0, +\infty)$ satisfying $f^{-1}(\{0\}) = \{0\}$ is metric-preserving if any one of the following conditions holds:
 (i) f is non-decreasing and sub-additive.
 (ii) f is concave.
 (iii) There exists constant $c > 0$ so that for any $x > 0$, $f(x) \in [c, 2c]$.

(3) [Urysohn's lemma]
Let (X, d) be a metric space, For any subset $A \subset X$, define
$$d_A : X \to [0, +\infty), \quad x \mapsto d_A(x) = \inf_{a \in A} d(x, a).$$
Prove:
(a) d_A is a continuous function on X.
(b) A is closed if and only if $d_A(x) = 0$ implies $x \in A$.
(c) (*Urysohn's lemma for metric spaces*) If A and B are closed subsets in (X, d) and $A \cap B = \emptyset$. Then there exists a continuous function $f : X \to [0, 1]$ such that
$$f \equiv 0 \text{ on } A, \qquad \text{and} \qquad f \equiv 1 \text{ on } B.$$

(4) [Uniform convergence as a metric convergence]
Let $f_n : (X, d_X) \to (Y, d_Y)$ $(n \in \mathbb{N})$ and $f : (X, d_X) \to (Y, d_Y)$ be maps between metric spaces.

(a) Define "uniform convergence": f_n converges uniformly to f on X if ...

(b) Suppose f_n are continuous, and converges to f uniformly. Prove: f is continuous.

(c) On the set $Y^X = \{f : X \to Y \mid f \text{ is any map}\}$, define

$$\bar{d}(f, g) := \sup_{x \in X} \frac{d_Y(f(x), g(x))}{1 + d_Y(f(x), g(x))}.$$

(i) Prove: $\bar{d}$ is a metric on Y^X.

(ii) Prove: f_n converges to f uniformly if and only if as elements in the metric space $(Y^X, \bar{d})$, f_n converges to f.