PROBLEM SET 2, PART 1: TOPOLOGY (H)
DUE: MARCH 07, 2022

(1) ["Uniform continuity" is not a topological conception]
Let (X, d_X) and (Y, d_Y) be metric spaces. We say a map $f : (X, d_X) \to (Y, d_Y)$ is *uniformly continuous* if

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

(a) Prove: $d_0(x, y) := |\arctan(x) - \arctan(y)|$ is a metric on $\mathbb{R}$.
(b) Prove: The metric d_0 and the absolute value metric $d(x, y) = |x - y|$ on $\mathbb{R}$ are topologically equivalent. Are they strongly equivalent?
(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map, i.e. $f(x) = x$. Is $f : (\mathbb{R}, d) \to (\mathbb{R}, d_0)$ uniformly continuous? Is $f : (\mathbb{R}, d_0) \to (\mathbb{R}, d)$ uniformly continuous? Conclude that "Uniform continuity" is not a topological conception.
(d) Is "uniform continuity" preserved if we replace metrics d_X, d_Y by strongly equivalent ones? Prove your conclusion.

(More generally, there is a structure called "uniform structure", which is a generalization of metric structure, so that one can define uniform continuous maps between spaces with uniform structures.For details, c.f. J.L. Kelley, *General Topology*.)

(2) [The product topology and product metrics]
(a) Prove Proposition 1.44 (the product topology is a topology).
(b) Let (X, d_X) and (Y, d_Y) be metric spaces. Endow the product space $X \times Y$ with the metric

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Prove:
(i) If U is open in (X, d_X), V is open in (Y, d_Y), then $U \times V$ is open in $(X \times Y, d_{X\times Y})$.
(ii) W is an open set in $(X \times Y, d_{X\times Y})$ if and only if for any $(x, y) \in W$, there exists $r > 0$ such that $B(x, r) \times B(y, r) \subset W$. [So the metric topology induced by the product metric is the same as the product topology induced by metric topologies.]

(c) (**NOT REQUIRED**) Prove: The same conclusion holds if we replace the metric $d_{X\times Y}$ above by

$$d^p_{X\times Y}((x_1, y_1), (x_2, y_2)) := (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{1/p},$$

where $1 \le p \le +\infty$. Note: for $p = \infty$ we define

$$d^\infty_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

(3) [Equivalence of neighborhoods axioms and open sets axioms: Proposition 1.37]

(a) Given a neighborhood structure $\mathscr{N}$ on X, one can define a topology $\mathscr{T}$ via

$$\mathscr{T} = \{U \subset X \ : \ U \in \mathscr{N}(x) \text{ for any } x \in U.\}$$

Check: $\mathscr{T}$ is a topology on X, i.e. it satisfies (O1)-(O3)

(b) Given a topology $\mathscr{T}$ on X, one can define, for any $x \in X$,

$$\mathscr{N}(x) = \{N \subset X : \exists U \in \mathscr{T} \text{ s.t. } x \in U \text{ and } U \subset N\}.$$

Check: $\mathscr{N}$ is a neighborhood structure on X, i.e. it satisfies (N1)-(N4).

(c) You may have already noticed that in doing part (a), you used only (N1)-(N3). Can we conclude that the set of axioms (N1)-(N3) is equivalent to the set of axioms (O1)-(O3)?

(d) (**NOT REQUIRED**) Prove: the set of axioms (N1)-(N4) is equivalent to the set of axioms (O1)-(O3). Namely, the process $\mathscr{T} \rightsquigarrow \mathscr{N}$ and $\mathscr{N} \rightsquigarrow \mathscr{T}$ described above are inverse to each other.

(4) [Furstenberg's topological proof of the infinitude of primes]

For any $a, b \in \mathbb{Z}$ with $b > 0$ we define

$$N_{a,b} := \{a + nb \mid n \in \mathbb{Z}\}.$$

(a) (**NOT REQUIRED**) Define a topology on $\mathbb{Z}$ by

$$\mathscr{T}_{Furs} = \{U \subset \mathbb{Z} \mid \text{ either } U = \emptyset, \text{ or } \forall a \in U, \exists b \in \mathbb{Z}_{>0} \text{ s.t. } N_{a,b} \subset U\}.$$

(i) Prove: $\mathscr{T}_{Furs}$ is a topology on $\mathbb{Z}$.

(ii) Prove: Each $N_{a,b}$ is open.

(iii) Prove: Each $N_{a,b}$ is closed. [Hint: $N_{a,b} = \mathbb{Z} \setminus \cup_{i=1}^{b-1} N_{a+i,b}$.]

(iv) Let $\mathscr{P} = \{2, 3, \cdots\}$ be the set of all prime numbers. Prove:

$$\mathbb{Z} \setminus \{1, -1\} = \cup_{p \in \mathscr{P}} N_{0,p}.$$

(v) Conclude that $\mathscr{P}$ is not a finite set. [Hint: the set $\{1, -1\}$ can't be open.]

(b) Define a function $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$d(a, b) = \begin{cases} 0, & a = b \\ 2^{-\tau(a-b)}, & a \neq b, \end{cases}$$

where $\tau(a - b)$ is the smallest positive integer that does not divide $a - b$.

(i) Prove: d is a metric on $\mathbb{Z}$.

(ii) Describe the metric balls $B(a, r)$.

(iii) Show that the metric topology generated by d is the topology $\mathscr{T}_{Furs}$ above.