PROBLEM SET 2, PART 2: TOPOLOGY (H)
DUE: MARCH 07, 2022

(1) [The Sorgenfrey line]
Endow $\mathbb{R}$ with the Sorgenfrey topology
$$\mathscr{T}_{Sorgenfrey} = \{U \subset \mathbb{R} \mid \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } [x, x+\varepsilon) \subset U\}.$$
- (a) Check: $\mathscr{T}_{Sorgenfrey}$ is a topology.
- (b) Prove: Every left-closed-right-open interval $[a, b)$ is both open and closed.
- (c) Prove: $\mathscr{T}_{Sorgenfrey}$ is strictly stronger than the usual topology $\mathscr{T}_{usual}$ on $\mathbb{R}$.
- (d) Explore the meaning of convergence in $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$.
- (e) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *right continuous* if $\lim_{x_n \to x_0+} f(x_n) = f(x_0)$. Prove: a function $f : \mathbb{R} \to \mathbb{R}$ is right continuous if and only if the map $f : (\mathbb{R}, \mathscr{T}_{Sorgenfrey}) \to (\mathbb{R}, \mathscr{T}_{usual})$ is continuous. [So people also call Sorgenfry topology *the right continuous topology.*]
- (f) **[Upper semi-continuous topology]** Let $(X, \mathscr{T})$ be any topological space. We say a function $f : X \to \mathbb{R}$ is *upper semi-continuous* at a point $x_0 \in X$ if for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $f(x) \le f(x_0) + \varepsilon$ holds for all $x \in U$, and we say f is an *upper semi-continuous* function if it is upper semi-continuous everywhere. Construct a new topology $\mathscr{T}_{u.s.c}$ on $\mathbb{R}$ so that a function $f : X \to \mathbb{R}$ is upper semi-continuous if and only if the map $f : (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{u.s.c})$ is continuous.

(2) [The pasting lemma]
Let X, Y be topological spaces. Consider a map $f : X \to Y$.
- (a) Suppose $X = A \bigcup B$, where A, B are both closed subsets in X. Suppose $f|_A : A \to Y$ and $f|_B : B \to Y$ are continuous. Prove: $f : X \to Y$ is continuous.
- (b) Show that the same result fails for $X = \bigcup_{n=1}^{\infty} A_n$, where each A_n is closed in X.
- (c) Prove: If $X = \cup_\alpha U_\alpha$, where each U_α is open in X, and if $f|_{U_\alpha} : U_\alpha \to Y$ is continuous, then $f : X \to Y$ is continuous.

(3) [Homeomorphisms]
- (a) Let $N = (0, \cdots, 0, 1)$ be the "north pole" of $S^n = \{(x^1, \cdots, x^{n+1}) | x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. Show that $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$ by explicitly construct a homeomorphism. [Hint: stereographic projection.]
- (b) Use Brouwer's invariance of domain theorem (see the end of Remark 1.58) to prove: If $n \neq m$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.
- (c) Prove: If $f : X \to Y$ is a homeomorphism, then for any $A \subset X$, $f : X \setminus A \to Y \setminus f(A)$ is a homeomorphism.
- (d) Let $\mathrm{Homeo}(X)$ be the set of all homeomorphisms from X to X. Prove: $\mathrm{Homeo}(X)$ is a group (with respect to the composition of maps). Moreover, if X and Y are homeomorphic, then the groups $\mathrm{Homeo}(X)$ and $\mathrm{Homeo}(Y)$ are isomorphic.

(4) (**NOT REQUIRED**) [Convergence in measure, almost everywhere convergence] Let X be the set of all bounded measurable functions defined on $[0,1]$. For any $f, g \in X$, we define

$$d(f,g) = \int_0^1 \min(|f(x) - g(x)|, 1)dx.$$

(a) Prove: d is a metric on X.
(b) Prove: $f_n \in X$ converges to f in measure if and only if f_n converges to f with respect to the metric d. (So in particular, "convergence in measure" is a topological convergence)
(c) Prove: almost everywhere convergence is not a topological convergence, i.e. there is no topology on X so that $f_n \to f$ a.e. if and only if $f_n \to f$ in that topology. [Hint: In real analysis, we learned that Riesz's theorem, which claims that if $f_n \to f$ in measure, then there is a subsequence $f_{n_k} \to f$ a.e.. Suppose there is such a topology. Find a sequence f_n in X that converges to f in measure, but fails to converge to f a.e.. Suppose such a topology exists. Since f_n fails to converge to f a.e., there is an open neighborhood U of f so that a subsequence sits outside U. But that subsequence still converges in measure, and thus has a sub-subsequence that converges a.e. to f, a contradiction.]