PROBLEM SET 3, PART 1: TOPOLOGY (H)
DUE: MARCH 14, 2022

(1) [Neighborhood basis]
Like a basis, we can define a *neighborhood basis* (or *neighborhood base*) as follows: A family $\mathcal{B}(x) \subset \mathcal{N}(x)$ of neighborhoods of x is called a *neighborhood basis at x* if for any $A \in \mathcal{N}(x)$, there exists $B \in \mathcal{B}(x)$ such that $B \subset A$.
   (a) Express $\mathcal{N}(x)$ in terms of a neighborhood basis $\mathcal{B}(x)$.
   (b) Define a conception of *neighborhood sub-basis*.
   (c) Write down a theorem that characterize the continuity of a map f at a point x via neighborhood basis and via neighborhood sub-basis, and prove your theorem.

(2) [Topologies on $\mathbb{R}^{\mathbb{N}}$]
Consider the space of sequences of real numbers,

$$X = \mathbb{R}^{\mathbb{N}} = \{(x_1, x_2, \cdots) \mid x_n \in \mathbb{R}\}.$$

On X we have defined three topologies: the box topology $\mathscr{T}_{box}$, the product topology $\mathscr{T}_{product}$, and the "uniform topology" $\mathscr{T}_{uniform}$ induced from the uniform metric

$$d_{uniform}((x_n), (y_n)) = \sup_{n \in \mathbb{N}} \min(|x_n - y_n|, 1).$$

   (a) Prove: $\mathscr{T}_{product} \subset \mathscr{T}_{uniform} \subset \mathscr{T}_{box}$.
   (b) One can also regard every element $(x_1, x_2, \cdots)$ in X as a map

$$f : \mathbb{N} \to \mathbb{R}, \quad n \mapsto x_n$$

   and thus identify X with the space of maps $\mathcal{M}(\mathbb{N}, \mathbb{R})$. Define the pointwise convergence topology $\mathscr{T}_{p.c.}$ on X, and prove $\mathscr{T}_{p.c.} = \mathscr{T}_{product}$.
   (c) Fix two elements $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ in X, and define a map

$$f : X \to X, \quad (x_1, x_2, \cdots,) \mapsto (a_1x_1 + b_1, a_2x_2 + b_2, \cdots).$$

   Prove that if we endow X with the product topology, then f is continuous. What if we endow X with the box topology?

(3) [Universality of the induced and co-induced topologies]
   (a) Prove Proposition 1.96.
   (b) Read page 38-39 on "co-induced topology" and prove Proposition 1.100.

(4) ["Product operation" for topologies is commutative and associative]
Let X_α ($\alpha \in \Lambda$) be topological spaces. Prove: For any decomposition $\Lambda = \bigcup_\beta \Lambda_\beta$ of the set of indices Λ (where $\Lambda_\beta \cap \Lambda_{\beta'} = \emptyset$ for $\beta \neq \beta'$), the product topological space $\prod_{\alpha\in\Lambda} X_\alpha$ is homeomorphic to the product topological space $\prod_\beta \left(\prod_{\alpha\in\Lambda_\beta} X_\alpha\right)$, where each product appeared above is endowed with the product topology.