PROBLEM SET 3, PART 2: TOPOLOGY (H)
DUE: MARCH 14, 2022

(1) [Embedding $\mathbb{RP}^2$ into $\mathbb{R}^4$]
Consider the map
$$f : S^2 \to \mathbb{R}^4, (x, y, z) \mapsto (y^2 - x^2, xy, xz, yz).$$
Prove: the image is homeomorphic to $\mathbb{RP}^2$.

(2) [Cone and suspension of S^n]
Prove the following by constructing a homeomorphism for each pair of spaces.
- (a) $C(S^n) \simeq B^{n+1}$.
- (b) $S(S^n) \simeq S^{n+1}$.
- (c) $B^n/S^{n-1} \simeq S^n$.

(3) [Quotient map v.s. open/closed map]
- (a) Suppose $p : X \to Y$ is a surjective continuous map. Prove: If p is either open or closed, then it is a quotient map.
- (b) Construct a quotient map that is neither open nor closed.
- (c) (Not required) Let $\mathrm{SO}(n)$ be the special orthogonal group. Define a map
$$f : \mathrm{SO}(n) \to S^{n-1}, \qquad A \mapsto Ae_1,$$
where $e_1 = (0, \cdots, 0, 1)$ is the "north pole vector" on S^{n-1}.
  - (i) Prove: f is surjective, continuous and open, and thus is a quotient map.
  - (ii) Consider the natural (right) action of $\mathrm{SO}(n-1)$ on $\mathrm{SO}(n)$ by
$$B \cdot A := A \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}, \qquad \forall B \in \mathrm{SO}(n-1), A \in \mathrm{SO}(n).$$
Prove: the orbits of this action are the fibers of the quotient map f.
  - (iii) Conclude that $\mathrm{SO}(n)/\mathrm{SO}(n-1) \simeq S^{n-1}$.

(4) [Covering space action]
Let G be a group acting on a topological space X. Let $Y = X/G$ be the orbit space, and $p : X \to Y$ be the quotient map. Let $U \subset X$ be an open set, such that
$$g \cdot U \cap U = \emptyset, \quad \forall g \neq e \in G.$$
Prove:
- (a) $V := p(U)$ is an open set in Y.
- (b) For any $g \in G$, the map $p_g = p \circ \tau_g : g^{-1} \cdot U \to V$ is a homeomorphism.