PROBLEM SET 4, PART 1: TOPOLOGY (H)
DUE: MARCH 21, 2022

(1) [“sequential conitnuous=continuous” for (A1) spaces]
Let X be an (A1) space, Y be any topological space. Prove: A map $f : X \to Y$ is continuous at x_0 if and only if it is *sequentially continuous* at x_0.

(2) [Locally finiteness]
Let $(X, \mathscr{T})$ be a topological space.
  (a) Let A, B be subsets in X. Prove: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
  (b) Let A_α be a family of subsets in X. Prove: $\cup_\alpha \overline{A_\alpha} \subset \overline{\cup_\alpha A_\alpha}$
  (c) Find an example so that $\cup_\alpha \overline{A_\alpha} \neq \overline{\cup_\alpha A_\alpha}$ for a family of subsets $A_\alpha \in \mathbb{R}$.
  (d) We say a family $\{A_\alpha\}$ of subsets in X is *locally finite* if for any $x \in X$, there exists an open neighborhood U_x of x so that $A_\alpha \cap U_x \neq \emptyset$ for only finitely many α's. Prove: If $\{A_\alpha\}$ is a *locally finite family*, then $\cup_\alpha \overline{A_\alpha} = \overline{\cup_\alpha A_\alpha}$.

(3) [Characterize continuity via interior]
In class we proved

> A map $f : X \to Y$ between two topological spaces is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ holds for any $A \subset X$.

Apply the idea of “open-closed” duality, write down the corresponding characterization of continuity of f via the interior operation, and then prove it.

(4) [Not required] [Convergence by net]
We call $(P, \preceq)$ a *directed set* if
- $(P, \preceq)$ is a partially ordered set (c.f. Def. 1.84),
- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

For a topological space X, a *net* is a map $f : (P, \preceq) \to X$ from a directed set $(P, \preceq)$ to X. We will use the notation (x_α) instead of a map “$f : \alpha \mapsto x_\alpha$” if there is no ambiguity. We say a net (x_α) *converges* to x_0, denoted by $x_\alpha \to x_0$, if for any neighborhood U of x, there is an $\alpha \in P$ such that $x_\beta \in U$ holds for any $\alpha \preceq \beta$.
  (a) Realize $\mathcal{N}(x)$ as a directed set. [You need to carefully choose the partial order relation so that it can be used in part (b) below.]
  (b) Prove: $x \in \overline{A}$ if and only if there exists a net (x_α) in A which converges to x_0.
  (c) Prove: A map $f : X \to Y$ is continuous if and only if for any net (x_α) in X which converges to a limit x_0, the net $(f(x_\alpha))$ in Y converges in Y to $f(x_0)$.