PROBLEM SET 4, PART 2: TOPOLOGY (H)
DUE: MARCH 21, 2022

(1) [Continuous maps from compact space to Hausdorff space]
Prove Lemma 2.1.20, Corollary 2.1.21 and Proposition 2.1.22.

(2) [Compactness for the "upper semi-continuous" topology]
In PSet2-2-1(f) you constructed the upper semi-continuous topology on $\mathbb{R}$,
$$\mathscr{T}_{u.s.c} = \{(-\infty, a) \mid a \in \mathbb{R}\}.$$
 (a) Is $(\mathbb{R}, \mathscr{T}_{u.s.c})$ compact? sequentially compact? limit point compact?
 (b) Describe all compact subsets in $(\mathbb{R}, \mathscr{T}_{u.s.c})$.
 (c) State a theorem called "the extremal value theorem for upper semi-continuous functions" and prove it.

(3) [Countably compact]
A topological space X is called *countably compact* if every countable open covering of X has a finite subcovering.
 (a) Prove: Closed subspace of a countably compact space is countably compact.
 (b) Prove: Any countably compact space is limit point compact.
 (c) Prove: X is countably compact if and only if it has the *nested sequence property*: for any nested sequence of non-empty closed sets $F_1 \supset F_2 \supset \cdots$, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.
 (d) Prove: Any sequentially compact space is countably compact.
 (e) Prove: The continuous image of a countably compact space is countably compact.

(4) [One point compactification]
Given any topological space $(X, \mathscr{T})$, we say a compact topological space Y is a *compactification* of X if there exists a homeomorphism $f : X \to f(X) \subset Y$ such that $\overline{f(X)} = Y$.
 (a) Prove: both S^1 and $[0, 1]$ are compactifications of $\mathbb{R}$.
 (b) For any non-compact topological space $(X, \mathscr{T})$, define a topology $\mathscr{T}^*$ on the set $X^* = X \cup \{\infty\}$ by
$$\mathscr{T}^* = \mathscr{T} \bigcup \{X^*\} \bigcup \{K^c \cup \{\infty\} \mid K \subset X \text{ is closed and compact.}\}.$$
 Prove: $\mathscr{T}^*$ is a topology on X^*, and $(X^*, \mathscr{T}^*)$ is a compactification of $(X, \mathscr{T})$. [This is called the *one-point compactification* of $(X, \mathscr{T})$.]
 (c) Prove: the one-point compactification of $\mathbb{N}$ is homeomorphic to $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ (as a subset in $\mathbb{R}$).
 (d) Construct a compact Hausdorff topology on any set X. [Hint: start with the discrete topology on $X \setminus \{x_0\}$]