PROBLEM SET 5, PART 1: TOPOLOGY (H)
DUE: MARCH 28, 2022

(1) [The topology of the Cantor set]
Recall that the Cantor set C is the following subset of $[0,1]$,

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right).$$

(a) Prove: Every point in the Cantor set is a limit point.
(b) Prove: As a subset of $[0,1]$, the Cantor set is nowhere dense.
(c) (Not required) For any closed subset $F \subset C$, prove: there exists a continuous map $f : C \to F$ so that $f(x) = x$ on F.
[Hints: F^c is the union of open intervals. Pick an element in each such interval that is not in C, and "push" points in the intervals to the "boundary points".]
(d) Define a map

$$g : \{0,1\}^{\mathbb{N}} \to [0,1], \quad a = (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{2a_k}{3^k}.$$

Prove: g induces a homeomorphism between $(\{0,1\}^{\mathbb{N}}, \mathcal{T}_{product})$ and C.
(e) (Not required) Show that there is continuous surjective map from C to $[0,1]^2$, by showing that

$$h : \{0,1\}^{\mathbb{N}} \to [0,1]^2, \quad a = (a_1, a_2, \cdots) \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{2^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{2^k} \right)$$

is continuous and surjective. Is h injective?

(2) [Sequentially compactness for products]
(a) Let $X_1, \cdots, X_n$ be sequentially compact topological spaces. Prove: the product space $X = X_1 \times \cdots \times X_n$ is sequentially compact.
(b) Is $X = \{0,1\}^{\mathbb{N}}$ sequentially compact when equipped with the box topology $\mathscr{T}_{box}$? Prove you claim.
(c) Now suppose (X_n, d_n) are compact metric spaces. Define a product metric on $X = \prod_{n=1}^{\infty} X_n$ via

$$d((x_n), (y_n)) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{(1 + \mathrm{diam}(X_k)) \cdot 2^n}.$$

Prove: The metric topology on X induced by d coincides with the product topology on X.

(3) [Compactness in order topology]
Let $(X, \leq)$ be a totally ordered set. For any subset $A \subset X$, we say $x \in X$ is a *least upper bound* of A if x is an upper bound of A, and there is no $x' < x$ which is an upper bound of A. Now endowed X with the order topology introduced in Definition 1.85. Prove: X is compact if and only if every subset (including the empty set $\emptyset$) of X has a least upper bound.
[Hints: X has a least upper bound implies that X has a maximal element. $\emptyset$ has a least upper bound implies that X has a minimal element. Try to prove that for any sub-base covering $\mathcal{U}$, there are $a < b$ so that $\{x|x < b\}$ and $\{x|x > a\}$ are elements in $\mathcal{U}$, and then apply Alexander subbase theorem.]

(4) [The existence of Banach limit](Not required)
Consider the vector space of all bounded sequences of real numbers,

$$X = l^{\infty} = \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{R} \text{ and } \sup_n |a_n| < \infty\}.$$

On X there is a naturally defined *shift* operator

$$T : X \to X, \qquad \{a_1, a_2, \cdots\} \mapsto \{a_2, a_3, \cdots\}.$$

A *mean* on X is a linear map $L : X \to \mathbb{R}$ such that

$$\inf a_n \leq L(\{a_n\}) \leq \sup a_n$$

holds for all $\{a_n\} \in X$. A *Banach limit* is a mean that is invariant under the shift operator T, i.e. such that $L(\{a_n\}) = L(T(\{a_n\}))$ holds for all $\{a_n\} \in X$.

(a) Define $L_m : X \to \mathbb{R}$ by $L_m(\{a_n\}) = \frac{1}{m}\sum_{i=1}^{m} a_i$. Prove: L_m is a mean for each m, and $\lim_{m\to\infty} |L_m(T(\{a_n\})) - L_m(\{a_n\})| = 0$.
(b) Let $\mathcal{M}$ be the set of all means on X. One can regard $\mathcal{M}$ as a subset of $\mathcal{M}(X, \mathbb{R}) = \mathbb{R}^X$, equipped with the product topology. Prove: $\mathcal{M}$ is compact.
[Hint: $\mathcal{M}$ is contained in $\prod_{\{a_n\}\in X}[\inf a_n, \sup a_n]$.]
(c) Prove: There exists a Banach limit on X.
[Hint: Compact implies limit point compact. Use (a).]
(d) What is the Banach limit of a convergent sequence? What is the Banach limit of $\{0, 1, 0, 1, 0, \cdots\}$?