PROBLEM SET 5, PART 2: TOPOLOGY (H)
DUE: MARCH 28, 2022

(1) [Completion of metric spaces]
Let X be a set, and (Y, d_Y) be metric spaces. Consider the space of bounded maps,

$$\mathcal{B}(X,Y) = \{f : X \to Y \mid f(X) \text{ is bounded in } Y\}$$

(a) Prove: the supremum metric $d_\infty(f,g) := \sup_{x\in X} d_Y(f(x), g(x))$ is a metric on $\mathcal{B}(X,Y)$.
(b) Prove: If Y is complete, so is $(\mathcal{B}(X,Y), d_\infty)$.

In what follows, suppose (X, d_X) is a metric space, and take $Y = \mathbb{R}$.

(c) Fix a point $x_0 \in X$. For any $a \in X$, define a function $f_a : X \to \mathbb{R}$ via $f_a(x) := d_X(x,a) - d_X(x, x_0)$. Prove: $f_a \in \mathcal{B}(X, \mathbb{R})$.
(d) Prove: the map

$$\Phi : (X, d) \to (\mathcal{B}(X,\mathbb{R}), d_\infty), a \mapsto f_a$$

is an isometric embedding, i.e. $d_X(a,b) = d_\infty(f_a, f_b)$ for any $a, b \in X$.
(e) Prove: Any metric space (X, d_X) admits a completion.
(f) (Not required) Prove: If (Y_1, d_1) and (Y_2, d_2) are two completions of (X, d_X), then (Y_1, d_1) and (Y_2, d_2) are isometric.

(2) [From limit point compact to sequentially compact]
In the proof of Proposition 2.3.25, we only used the following two properties:

- Every $x \in X$ has a *descending* countable neighborhood basis $U_1^x \supset U_2^x \supset \cdots$.
- If x is a limit point of A, then every neighbourhood of x contains infinitely many points of A.

As a consequence, there are many other topological spaces in which limit point compact is equivalent to sequentially compact:

(a) Prove Proposition 2.3.26.
(b) Prove that in Proposition 2.3.26, one can weaken the Hausdorff condition to the following (T1) condition:
    (T1): For any $x \neq y$ in X, there exists open sets U and V in X so that $x \in U \setminus V$ and $y \in V \setminus U$.
(c) The (T1) condition is equivalent to a sentence on page 1 of today's notes. Find out it and prove the equivalence.

(3) [Closed unit ball in l^2]
Consider the metric space l^2 given in Example 1.6(3).

(a) Prove: l^2 is complete.
(b) Prove: The closed unit ball $\overline{B(0,1)}$ and the unit sphere $S(0,1)$ are non-compact.
(c) Prove: If $K \subset l^2$ is compact, then K has no interior point.

(4) [Lebesgue property]
We say a metric space (X, d) has the Lebesgue property if any open covering of X has a positive Lebesgue number.

(a) Look at our proof of "sequentially compact $\Longrightarrow$ compact" in the proof of Theorem 2.3.28. What did we really proved? Your answer should be of the form ["condition A"+"condition B" implies compactness], and thus we have a new characterization of compactness in metric space.

(b) Prove: If (X, d_X) has the Lebesgue property, then it is complete.

(c) Prove: (X, d_X) has the Lebesgue property if and only if for any metric space (Y, d_Y), any continuous map $f : X \to Y$ is uniformly continuous.

(d) Suppose (X, d_X) has the Lebesgue property. Prove: If A, B are non-empty disjoint closed subsets in (X, d), then $dist(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\} > 0$.