PROBLEM SET 6, PART 1: TOPOLOGY (H)
DUE: APRIL 6, 2022

(1) [More on LCH]
   (a) [Structure of noncompact LCH]
      (i) Let K be a compact Hausdorff space, $p \in K$ and $X = K \setminus \{p\}$ is non-compact. Prove: X is LCH.
      (ii) Conversely, suppose X be a non-compact LCH. Let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. Prove: X^* is compact and Hausdorff.
   (b) [The evaluation map could fail to be continuous without local compactness] Consider the evaluation map

$$e : \mathbb{Q} \times \mathcal{C}(\mathbb{Q}, [0, 1]) \to [0, 1], \quad (x, f) \mapsto e(x, f) = f(x).$$

      (i) Prove: $\mathbb{Q}$ is not locally compact.
      (ii) Prove: for any $q_1 \in \mathbb{Q}$ and any closed subset $A \subset \mathbb{Q}$ with $q_1 \notin A$, there is a continuous function $f_1 \in \mathcal{C}(\mathbb{Q}, [0, 1])$ such that $f(q_1) = 1, f(A) = 0$.
      (iii) (Not required) Now let $f_0 \in \mathcal{C}(\mathbb{Q}, [0, 1])$ be the zero map $f_0(\mathbb{Q}) = 0$, and take any $q_0 \in \mathbb{Q}$. Prove: e is not continuous at (q_0, f_0) (where we endow $\mathcal{C}(\mathbb{Q}, [0, 1])$ with the compact convergence topology).
      [Hint: For any open neighborhood U of q_0 and any compact set K in $\mathbb{Q}$, there exists $q_1 \in U \setminus K$. Construct a continuous function using (b).]

(2) [More on compact-open topology]
   (a) Prove Proposition 2.4.22, i.e. (Y, d) is a metric space, then $\mathscr{T}_{c.c.} = \mathscr{T}_{c.o.}$.
   (b) Prove Proposition 2.4.23, i.e. if Y s LCH, then the composition map is continuous with respect to $\mathscr{T}_{c.o.}$.
   (c) Prove: If X is locally compact and Hausdorff, then

$$S(\{x\}, U) = \bigcup_{\text{compact neighborhood } K \text{ of } x} S(K, U).$$

   [Hint for (b) and (c): Use Proposition 2.4.16]

(3) [Compactly generated spaces]
   (a) Read the materials on compactly generated spaces (page 99), and prove: any locally compact space is compactly generated.
   (b) Prove: Any first countable space is compactly generated.
   (c) Find a compactly generated space that is not locally compact. [Hint: PSet5-2]
   (d) Let $(X, \mathscr{T})$ be any topological space. Prove: there exists a topology $\mathscr{T}' \supset \mathscr{T}$ such that $(X, \mathscr{T}')$ is compactly generated, and a set is compact with respect to $\mathscr{T}'$ if and only if it is compact with respect to $\mathscr{T}$.
   [Hint: Construct topology by needs!]

(4) [Applications of Arzela-Ascoli]

(a) Suppose $k = k(x, y) \in \mathcal{C}([0, 1] \times [0, 1], \mathbb{R})$. For any $f \in \mathcal{C}([0, 1], \mathbb{R})$, define

$$Kf(x) = \int_0^1 k(x, y)f(y)dy.$$

Prove: K is a *compact operator*, i.e. it maps any bounded subset in $(\mathcal{C}([0, 1], \mathbb{R}), d_\infty)$ into a compact subset in the same space.

(b) (Not required) We want to minimize the functional $\Phi[f] := \int_{-1}^1 f(t)dt$. Consider the set

$$\mathcal{F} = \{f \in \mathcal{C}([-1, 1], [0, 1]) \mid f(-1) = f(1) = 1\}.$$

(i) What is $\inf_{f\in\mathcal{F}} \Phi[f]$? Is the infimum attained?

(ii) For any constant $C > 0$, let

$$\mathcal{F}_C = \{f \in \mathcal{F} \mid |f(x) - f(y)| \leq C|x - y|\}.$$

Prove: The infimum $\inf_{f\in\mathcal{F}_C} \Phi[f]$ is attained. Can you find the function?

(c) (Not required) Prove Theorem 2.5.12.