PROBLEM SET 6, PART 2: TOPOLOGY (H)
DUE: APRIL 06, 2022

(1) [Topological algebra]
Let X be a topological space. Endow $\mathcal{C}(X,\mathbb{R})$ with the compact convergence topology.
  (a) Prove: The addition, multiplication and the scalar multiplication
$$a : \mathcal{C}(X,\mathbb{R}) \times \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R}), \quad (f,g) \mapsto a(f,g) = f + g,$$
$$m : \mathcal{C}(X,\mathbb{R}) \times \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R}), \quad (f,g) \mapsto m(f,g) = fg,$$
$$s : \mathbb{R} \times \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R}), \quad (\lambda, g) \mapsto s(\lambda, f) = \lambda g$$
are continuous. (As a consequence, $\mathcal{C}(X,\mathbb{R})$ is a topological algebra.)
  (b) Prove Proposition 2.6.4 (the closure of a subalgebra of topological algebra is a closed subalgebra).

(2) [Applications of Stone-Weierstrass]
  (a) Prove: Any continuous function on $[0,1]$ can be approximated uniformly by functions of the form
$$a_0 + a_1 e^x + a_2 e^{2x} + \cdots + a_n e^{nx}, \quad n \in \mathbb{N}.$$
  - As a consequence, prove if f is a continuous function on $[0,1]$ satisfying

(*) $$\int_0^1 f(x) e^{nx} dx = 0, \quad n = 0, 1, 2 \cdots,$$

then $f = 0$.
  - What if (*) holds only for even n?

  (b) Let X, Y be compact Hausdorff spaces. Prove: any $f \in \mathcal{C}(X \times Y, \mathbb{R})$ can be approximated uniformly by functions of the form
$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_n(x)g_n(y), \quad n \in \mathbb{N},$$
where $f_k \in \mathcal{C}(X,\mathbb{R}), g_k \in \mathcal{C}(Y,\mathbb{R})$.

(3) [Stone-Weierstrass for complex-valued functions]
  (a) Prove Theorem 2.6.16 (Stone-Weierstrass for complex-valued functions).
  (b) Prove: Any complex-valued continuous function on $S^1 = \mathbb{R}/\mathbb{Z}$ can be approximated uniformly by functions of the form
$$\sum_{k=-n}^{n} a_k e^{-2\pi i k x}, \quad n \in \mathbb{N}.$$

(4) [Stone-Weierstrass on LCH] (Not requried)
   (a) Let X be LCH. Prove: $\mathcal{C}_0(X, \mathbb{R})$ is an algebra.
   (b) Prove Theorem 2.6.17 (Stone-Weierstrass theorem on LCH).
   (c) Prove: Any $f \in C_0([0, +\infty), \mathbb{R})$ can be approximated uniformly by functions of the form

$$\sum_{k=-n}^{n} a_k e^{-kx}, \quad n \in \mathbb{N}.$$