PROBLEM SET 7, PART 1: TOPOLOGY (H)
DUE: APRIL 11, 2022

(1) [Lindelöff Property]
  (a) Prove Proposition 2.7.13.
  (b) Prove Proposition 2.7.14.
  (c) Check: $(\mathbb{R}, \mathscr{T}_{cocountable})$ is Lindelöf but not σ-compact.
  (d) Check: The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$ is Lindelöf.

(2) [The Sorgenfrey plane]
Consider the product of two Sorgenfrey lines,

$$(\mathbb{R}^2, \mathscr{T}_{Sorgenfrey}) := (\mathbb{R}, \mathscr{T}_{Sorgenfrey}) \times (\mathbb{R}, \mathscr{T}_{Sorgenfrey}),$$

which is known as the *Sorgenfrey plane*.
  (a) Prove: It is first countable, separable but not second countable.
  (b) Prove: Is it Hausdorff?
  (c) Consider the subspace $A = \{(x, -x) \mid x \in \mathbb{R}\}$. Is it closed? What is the induced subspace topology on A?
  (d) Prove: It is not Lindelöf.

(3) [Closedness of graph]
Let X, Y be topological spaces, define the *graph* of a map $f : X \to Y$ to be the set

$$G_f := \{(x, f(x) \mid x \in X\} \subset X \times Y.$$

  (a) Prove: Y is Hausdorff $\iff$ for any X and $f \in \mathcal{C}(X, Y)$, G_f is closed in $X \times Y$.
  (b) Construct a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed.
  (c) Prove: If Y is compact Hausdorff, then f is continuous $\iff$ G_f is closed.

(4) [Hereditary properties]
A topological property P is called *hereditary* if

$$\boxed{(X, \mathscr{T}) \text{ satisfies } P \Longrightarrow \text{ Any subspace } Y \text{ of } X \text{ satisfies } P.}$$

  (a) Prove: (A1) and (A2) are hereditary, but (T4) is not hereditary.
  [Hint: Given any $(X, \mathscr{T})$, consider $(X \cup \{\infty\}, \mathscr{T} \cup \{X \cup \{\infty\}\})$]
  (b) Which of the following properties are hereditary:
  compact/sequentially compact/locally compact/separable/Lindelöf/(T1)/(T2)/(T3)
  (c) A topological property P is called *closed hereditary* if

$$\boxed{(X, \mathscr{T}) \text{ satisfies } P \Longrightarrow \text{ Any closed subspace } Y \text{ of } X \text{ satisfies } P.}$$

  For those non-hereditary properties above, determine whether they are closed hereditary.