# PROBLEM SET 7, PART 2: TOPOLOGY (H)
## DUE: APRIL 11, 2022

(1) [Productive properties]
A topological property $P$ is called *productive* if

$$\boxed{\text{Each } (X_\alpha, \mathscr{T}_\alpha) \text{ satisfies } P \Longrightarrow (\prod_\alpha X_\alpha, \mathscr{T}_{prodcut}) \text{ satisfies } P.}$$

(a) Prove: (T1), (T2) and (T3) are productive.
(b) Conversely, if $(\prod_\alpha X_\alpha, \mathscr{T}_{prodcut})$ is (T1), (T2) or (T3), can we conclude that each $(X_\alpha, \mathscr{T}_\alpha)$ is (T1), (T2) or (T3)?
(c) Is (T4) productive? Is Lindelöf productive?
(d) Prove: *separable* and *metrizable* are not productive. What about (A1), (A2)?
(e) Can you introduce a weaker version of productivity, so that those non-productive properties in part (d) satisfy the weaker one?

(2) [Baire space]
A topological space is called a *Baire space* if every intersection of a countable collection of open dense sets in the space is also dense.
(a) Use "open-closed" duality to give an equivalent characterization of Baire space.
(b) Prove: Any complete metric space is a Baire space.
(c) Prove: Any compact Hausdorff space is a Baire space.
(d) Prove: Any locally compact Hausdorff space is a Baire space.

(3) [Applications of Urysohn lemma]
(a) Let $X$ be a compact Hausdorff space, $x_0 \in X$, and $U$ is an open neighborhood of $x_0$. Prove: For any $\varepsilon > 0$ and any continuous function $f : X \to \mathbb{R}$, there exists a continuous function $g : X \to \mathbb{R}$ satisfying all of the following three conditions:
- $\sup_{x\in X} |g(x) - f(x)| < \varepsilon$.
- $g = f$ on $U^c$.
- there exists a neighborhood $V$ of $x_0$ such that $g(x) \equiv f(x_0)$ on $V$.

(b) Let $X$ be LCH. Recall
- $\mathcal{C}_b(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and bounded}\}$.
- $\mathcal{C}_c(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and compactly supported}\}$.
- $\mathcal{C}_0(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and vanishes at infinity}\}$.

On $\mathcal{C}_b(X, \mathbb{R})$ we have a metric $d_\infty(f, g) := \sup_{x\in X} |f(x) - g(x)|$. Prove: The closure of $\mathcal{C}_c(X, \mathbb{R})$ in $\mathcal{C}_b(X, \mathbb{R})$ is $\mathcal{C}_0(X, \mathbb{R})$.

(4) [Locally metrizable] (Not required)
A topological space $X$ is said to be *locally metrizable* if for any $x \in X$, there is a neighborhood $U$ of $x$ that is metrizable. Prove: If $X$ is compact Hausdorff, then $X$ is metrizable if and only if it is locally metrizable.
[Hint: Cover $X$ by finitely many compact metrizable neighborhood.]