PROBLEM SET 8, PART 1: TOPOLOGY (H)
DUE: APRIL 18, 2022

(1) [Uniqueness of extension]
Let X, Y be topological spaces, $A \subset X$ be a *dense subset*, and $f : A \to Y$ be a continuous map.
  (a) Prove: If Y is a (T2) space, then f admits at most one continuous extension.
  (b) Does the same conclusion hold if Y is a (T1) space? If yes, prove it; if no, give a counterexample.

(2) [Tietze extensions with restrictions]
Let $(X, \mathscr{T})$ be a (T4) space, $A \subset X$ be closed.
  (a) Let $f : A \to \mathbb{C}$ be a continuous complex-valued function with
$$|f(x)| \le 1, \quad \forall x \in A.$$
Prove: f can be extended to a continuous function $\tilde{f} : X \to \mathbb{C}$ so that
$$|\tilde{f}(x)| \le 1, \quad \forall x \in X.$$
  (b) Let $f : A \to \mathbb{R}$ and $g_1, g_2 : X \to \mathbb{R}$ be continuous functions, and suppose
$$g_1(x) \le f(x) \le g_2(x),\ \forall x \in A \qquad \underline{\text{and}} \qquad g_1(x) \le g_2(x),\ \forall x \in X.$$
Prove: f can be extended to a continuous function $\tilde{f} : X \to \mathbb{R}$ such that
$$g_1(x) \le \tilde{f}(x) \le g_2(x), \quad \forall x \in X.$$

(3) [Retraction]
Let X be a topological space, $A \subset X$ be a subspace. We say A is a *retract* of X if there exists a continuous map $r : X \to A$ such that
$$r(x) = x, \quad \forall x \in A.$$
Such a map r is called a *retraction*.
  (a) Prove: If X is Hausdorff, A is a retract of X, then A is closed.
  (b) Prove: A is a retract of X if and only if for any topological space Y, any continuous map $f : A \to Y$ has an extension $\tilde{f} : X \to Y$.
  (c) Suppose X is normal and A is closed. Prove: If Y is a retract of $\mathbb{R}^J$ (with product topology, where J is any set), then any continuous map $f : A \to Y$ admits a continuous extension $\tilde{f} : X \to Y$.

(4) [Different compactifications][Not required]
Let X, Y, Z be LCH spaces.
  (a) Construct at least three different compactifications of the plane $\mathbb{R}^2$.

(b) Prove that the Cech-Stone compactification βX is the largest compactification of X: For any compact Hausdorff compatification K of X (with an embedding $\varphi : X \to K$), there is a surjective continuous closed map $F : \beta X \to K$ which extends the embedding $\varphi : X \to K$.

(c) Similarly, prove that the one point compactification X^* is the smallest compactification of X.

(d) Given any continuous map $\varphi : X \to Y$, we constructed a continuous map $\beta\varphi : \beta X \to \beta Y$. Prove that the "lifting" $\varphi \rightsquigarrow \beta\varphi$ is "functorial" in the following sense:

(i) If $\mathrm{Id_X}$ is the identity map, then $\beta\mathrm{Id_X} = \mathrm{Id}_{\beta X}$.

(ii) If $\varphi : X \to Y$, $\psi : Y \to Z$ be continuous maps, then $\beta(\psi \circ \varphi) = \beta\psi \circ \beta\varphi$.