# PROBLEM SET 8, PART 2: TOPOLOGY (H)
## DUE: APRIL 18, 2022

(1) [Products of paracompact spaces]
   (a) Prove: The Sorgenfrey line is paracompact, while the Sorgenfrey plane is not. [Hint: The Sorgenfrey plane is not normal.]
   (b) Is paracompactness productive? Is it preserved under continuous maps?
   (c) Prove: If $X$ is compact, $Y$ is paracompact, then $X \times Y$ is paracompact.

(2) [LCH version of P.O.U.]
Let $X$ be a locally compact, $\sigma$-compact, Hausdorff space, and $\mathscr{U} = \{U_\alpha\}$ is an open cover of $X$. Prove:
   (a) There exits two locally finite open coverings $\mathscr{V} = \{V_n\}$ and $\mathscr{W} = \{W_n\}$ such that
      - $W_n \subset \overline{W_n} \subset V_n \subset \overline{V_n}$, and $\overline{V_n}$ is compact,
      - For each $n$, there exists $U_\alpha \in \mathscr{U}$ such that $\overline{V_n} \subset U_\alpha$.
   (b) Prove Theorem 2.10.15(LCH version of P.O.U.).

(3) [Examples and non-Examples of topological manifolds]
   (a) Prove: Every topological manifold is $\sigma$-compact.
   (b) Prove: $\mathbb{RP}^n$ is a topological manifold.
   (c) (line with doubled point) Let $X = (\mathbb{R} \times \{0,1\})/\sim$, where $(x,0) \sim (x,1)$ for all $x \neq 0$. Prove: $X$ is (A2) and locally Euclidian, but not (T2).
   (d) [NOT Required] (long line) Let $\Omega$ be the smallest uncountable well-ordered set. That is, $\Omega$ is an uncountable set, and there is a well-order $<$ on $\Omega$ such that for any $a \in \Omega$, the set $\{b \in \Omega \mid b < a\}$ is countable. Let $L = \Omega \times [0,1)$. Define an order on $L$ via
$$(a,t) \prec (b,s) \text{ if and only if “} a < b \text{” or “} a = b \text{ and } t < s\text{”}.$$
   For any $x \prec y$ in $L$, we define $(x,y) = \{z \in L \mid x \prec z \prec y\}$.
      (i) Prove: These “intervals” $(x,y)$ form a basis of a topology on $L$.
      (ii) Prove: With respect to this topology, $L$ is (T2), locally Euclidean but not (A2). It is called the *long line*.
      [Hint: By the definition of well-order, for any $a \in \Omega$, the set $\{b \in \Omega \mid a < b\}$ has a minimal element, called the successor of $a$. Define charts on $L$ by
$$\varphi : \{a\} \times (0,1] \cup \{a'\} \times (0,1) \to (-1,1), \quad \varphi(a,t) = t - 1 \text{ and } \varphi(a',t) = t.$$
      where $a'$ is the successor of $a$.]

(4) [An application of P.O.U.]
Let $X$ be Hausdorff and paracompact, $f : X \to \mathbb{R}$ be lower semi-continuous and $g : X \to \mathbb{R}$ be upper semi-continuous. Moreover, assume $f(x) > g(x), \forall x \in X$. Prove: there exists a continuous function $h : X \to \mathbb{R}$ such that $f(x) > h(x) > g(x), \forall x \in X$.