PROBLEM SET 9, PART 1: TOPOLOGY (H)
DUE: MAY 04, 2022

(1) [Connectedness of subspace]
Let $(X, \mathscr{T})$ be a topological space, and $Y \subset X$ be a subspace. Which of the following statements are equivalent to the fact

$$\boxed{Y \text{ is disconnected}}\ ?$$

Prove the equivalence for the correct ones and give counterexamples for the wrong ones:

(a) There exists non-empty sets $A, B \subset X$ with $A \cap \overline{B} = \overline{A} \cap B = \emptyset$, such that $Y = A \cup B$, where the closure is taken in to be the closure in X.
(b) There exists open sets A, B in X with $A \cap B \cap Y = \emptyset$, such that $Y \subset A \cup B$ and $A \cap Y \neq \emptyset, B \cap Y \neq \emptyset$.
(c) There exists disjoint open sets A, B in X with $A \cap Y \neq \emptyset, B \cap Y \neq \emptyset$, such that $Y \subset A \cup B$.
(d) There exists disjoint closed sets A, B in X with $A \cap Y \neq \emptyset, B \cap Y \neq \emptyset$, such that $Y \subset A \cup B$.
(e) There exists a set A which is both open and closed in X such that $A \cap Y \neq \emptyset$ and $A \cap Y \neq Y$.
(f) There is a surjective continuous map $f : Y \to \{0, 1\}$.

(2) [Connected components]
Let X be a topological space. The *connected component* containing $x \in X$ is defined to be the maximal connected subsets of X containing x.

(a) Prove: The connected component containing x is the union of all connected subsets of X that contains x.
(b) Prove: Each connected component is a closed subset.
(c) Give an example showing that the connected component need not be open.
(d) (Generalization of Proposition 3.1.8) Prove: If $f : X \to Y$ is continuous, then for any subset A of X, the cardinality of connected components of $f(A)$ is no more than the cardinality of connected components of A.
(e) (Generalization of Proposition 3.1.18) Denote the connected component of X_α containing x_α to be $C(x_\alpha)$. Prove: the connected component of $\prod_\alpha X_\alpha$ containing the point (x_α) is $\prod_\alpha C(x_\alpha)$.

(3) [Non-homeomorphic spaces]

(a) Show that the following spaces are pairwise non-homeomorphic:

$$\mathbb{R}, \quad \mathbb{Z}, \quad S^1, \quad \mathbb{R}^2, \quad [0,1], \quad [0,1)$$

(b) Consider

$$A = (0,1) \cup \{2\} \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \cdots,$$
$$B = (0,1] \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \cdots.$$

Prove: There exists continuous bijection $f : A \to B$ and continuous bijection $g : B \to A$, however, A and B are not homeomorphic.

[You may compare this with Cantor–Schröder–Bernstein theorem in set theory]

(4) [Connected + suitable separation axioms v.s. countability]

(a) Prove: If $(X, \mathscr{T})$ is (T1), (T4) and connected, and X contains at least two elements, then X contains uncountably many elements.

(b) Can we replace (T4) by (T3)?

(c) [Not required][The Golomb space] Define a topology on $\mathbb{N}_{>0}$ as follows: For any coprime positive integers a and b, let $D_{a,b} = \mathbb{N}_{>0} \cap \{a + bk \mid k \in \mathbb{N}_{\geq 0}\}$. Consider the topology $\mathscr{T}_{Golomb}$ generated by these $D_{a,b}$'s. It turns out that $(\mathbb{N}_{>0}, \mathscr{T}_{Golomb})$ is (T2), connected but contains countably elements:

(i) Prove: $\mathcal{B} = \{D_{a,b} \mid a, b \text{ are coprime positive integers}\}$ is a basis of $\mathscr{T}_{Golomb}$.

(ii) Prove: $(\mathbb{N}_{>0}, \mathscr{T}_{Golomb})$ is (T2).

(iii) Prove: $(\mathbb{N}_{>0}, \mathscr{T}_{Golomb})$ is connected. Is it compact or (T3) or metrizable? [In proving connectedness, you may need the following consequence of *Chinese remainder theorem* from number theory: If b_1 and b_2 are coprime, then $D_{a_1,b_1} \cap D_{a_2,b_2} \neq \emptyset$.]

(iv) The *Dirichlet Theorem* in number theory asserts that every $D_{a,b}$ (with a, b coprime) contains infinitely many prime numbers. Explain this using the language of topology.