# PROBLEM SET 9, PART 2: TOPOLOGY (H)
# DUE: MAY 04, 2022

(1) [Path connectedness: examples]
  (a) Although looks quite non-obvious, the set $\mathbb{R}^2 - \mathbb{Q}^2$ is path-connected. We give two proofs here:

  *First proof.* Since $\mathbb{Q}^2$ is a countable set, for any $x \in \mathbb{R}^2 - \mathbb{Q}^2$, there exist uncountably many lines $l$ s.t.

$$x \in l \subset \mathbb{R}^2 - \mathbb{Q}^2.$$

  Now for $x \neq y \in \mathbb{R}^2 - \mathbb{Q}^2$, pick two such lines, one contains $x$ and the other contains $y$, such that they are not parallel. Now you can connect $x$ to the intersection point through the first line, then to $y$ through the second line. □

  *Second proof.* Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 - \mathbb{Q}^2$. If $x_1, x_2 \in \mathbb{R} - \mathbb{Q}$, then we pick $y_0 \in \mathbb{R} - \mathbb{Q}$, and connect $(x_1, y_1)$ to $(x_1, y_0)$ through the line $x = x_1$, and connect $(x_1, y_0)$ to $(x_2, y_0)$ through the line $y = y_0$, and finally connect $(x_2, y_0)$ to $(x_2, y_2)$ through the line $x = x_2$. Similar arguments holds if $x_1, y_2 \in \mathbb{R} - \mathbb{Q}$ or $y_1, y_2 \in \mathbb{R} - \mathbb{Q}$ or $x_2, y_1 \in \mathbb{R} - \mathbb{Q}$. □

  It turns out that each proof can be extended to prove a more general result on path-connectedness:

  **Proposition 0.1.** *Let $S \subset \mathbb{R}^n$ be ... then $\mathbb{R}^n - S$ is path connected.*

  **Proposition 0.2.** *Let $X, Y$ are path-connected, and ...*

  Complete the full statements.

  (b) Show that the topological space

$$(X = \{v, s\}, \mathscr{T} = \{\emptyset, \{s\}, \{v, s\}\}$$

  is path-connected.

(2) [Locally connectedness]
  (a) Define the conception:

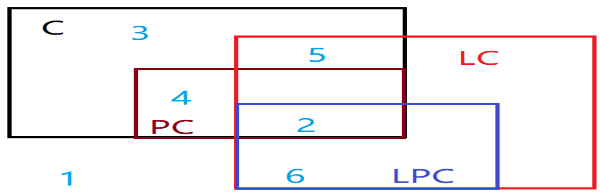
  **Definition 0.3.** We say a topological space $X$ is <u>locally connected</u> if ......

  (b) Consider $(\mathbb{R}, \mathscr{T}_{cocountable})$. Is it connected? locally connected? path connected? locally path connected?

  (c) For simplicity, let's denote

  C=connected, LC=locally connected,
  PC=path connected, LPC=locally path connected.

  Give examples in region 1-6 for the following picture: (the remaining two parts are more complicated. You can try if you want to challenge yourself..)




(d) Prove: If $X$ is compact and locally connected, then $X$ has finitely many connected components. Can we remove the locally connectedness condition?

(e) Prove: $X$ is locally connected if and only if for any open set $U$ in $X$, any connected component of $U$ is open.
(In particular, any connected component of a locally connected space is open.)

(f) (Not required) Suppose $X$ is locally connected, $f : X \to Y$ is continuous. Prove: if $f$ is either open or closed, then $f(X)$ is locally connected.
Can we remove the assumption "$f$ is either open or closed"?

(3) [Components and path components]

(a) Find the components and path component for the following spaces:
 (i) The Sorgenfrey line.
 (ii) $(\mathbb{R}, \mathscr{T}_{cocountable})$.
 (iii) $(\mathbb{R}^{\mathbb{N}}, \mathscr{T}_{uniform})$.

(b) Prove Proposition 3.2.22 and Proposition 3.2.23, namely, $\pi_0$ and $\pi_c$ are functors.

(4) (Not required) [Components of topological groups]
Let $G$ be a topological group.

(a) Prove: For any normal subgroup $N$ of $G$, the quotient group $G/N$ is a topological group.

(b) Prove: $\pi_0(G)$, $\pi_c(G)$ are both topological groups. What's the relation between these two groups?

(c) Are $\pi_0(G)$ and $\pi_c(G)$ Hausdorff spaces?

(d) Find the relations between $\pi_0(G_1 \times G_2)$ and $\pi_0(G_1), \pi_0(G_2)$, where $G_1, G_2$ are topological groups.