PROBLEM SET 10, PART 1: TOPOLOGY (H)
DUE: MAY 9, 2022

(1) [Constructing homotopies]
  (a) Prove Proposition 3.3.3 (composition, pull-back and push-forward)
  (b) Prove that "homotopy equivalence between topological spaces" is an equivalence relation (Remark 3.3.9(3)).
  (c) Prove Proposition 3.3.17(1) and (3).

(2) [Maps to S^n]
  (a) Prove: Any non-surjective continuous map $f : X \to S^n$ is null-homotopic.
  (b) Let $f, g : X \to S^n$ be continuous maps. Suppose they are never anti-podal, i.e. $g(x) \neq -f(x)$ holds for all x. Prove: f is homotopic to g.
  (c) Let $\overline{B^{n+1}}$ be the closed unit ball in $\mathbb{R}^{n+1}$. Prove: There exists a retraction $f \in \mathcal{C}(\overline{B^{n+1}}, S^n)$ if and only if Id_{S^n} is null-homotopic.
  [Hint: For "only if" part, use the fact $\overline{B^{n+1}}$ is convex; for "if" part, use the fact "$\overline{B^{n+1}}$ is the cone over S^n".]

(3) [Deformation retract]
We say A is a *weak deformation retract* of X if there exists a retraction $r : X \to A$ so that Id_X is homotopic to $\iota \circ r : X \to X$, where $\iota : A \hookrightarrow X$ is the inclusion map. In other words, A is a *weak deformation retract* of X if there exists a continuous map (called a *weak deformation retraction*) $F : [0, 1] \times X \to X$ such that

$$F(0, x) = x, F(1, x) \in A, \ \forall x \in X \quad \text{and} \quad F(1, a) = a, \forall a \in A.$$

A weak deformation retraction F is called a *strong deformation retraction* if

$$F(t, a) = a, \quad \forall a \in A, \forall t \in [0, 1].$$

[In some books, people call weak deformation retract defined above a deformation retract, while in some other books (includes Munkres's book and Hatcher's book) people call strong deformation retract defined above a deformation retract.]
  (a) Construct a strong deformation retraction $\mathbb{R}^{n+1} \setminus \{0\}$ to S^n.
  (b) Construct a strong deformation retraction from $\mathbb{T}^2 - \{pt\}$ (i.e. the torus with one point removed) $S^1 \vee S^1$ (i.e. "figure 8").
  (c) Prove: If $A \subset X$ is a weak deformation retract, then $A \sim X$.
  (d) (Not required)[Compare with Exercise for Section 2.9] Prove: $A \subset X$ is a weak deformation retract of X if and only if it satisfies the following two properties:
    - For any topological space Y, any continuous map $f : A \to Y$ has a continuous extension $\tilde{f} : X \to Y$.
    - For any topological space Y and any continuous maps $f, g : X \to Y$, if $f|_A$ is homotopic to $g|_A$, then f is homotopic to g.

(4) [Contractible spaces]

(a) Prove that the following are equivalent:

(i) X is contractible.

(ii) X is homotopy equivalent to a point.

(iii) X weak deformation retracts to a point. [However, there are examples of topological spaces that are contractible but do not strong deformation retract to any point(c.f. Hatcher, Algebraic Topology, Exercise 0.6).]

(b) Recall that the (topological) cone $C(X)$ of any space topological space X is

$$C(X) = X \times [0,1]/X \times \{0\}.$$

(i) Prove: For any X, the topological cone $C(X)$ is contractible.

(ii) (Not required) Let Y be any topological space, and $f \in \mathcal{C}(X, Y)$ be a continuous map. Prove: f is null-homotopic if and only if f has a continuous extension $\hat{f} : C(X) \to Y$.

(c) Suppose Brouwer's fixed point theorem holds, i.e. any continuous map $f : \overline{B^n} \to \overline{B^n}$ has a fixed point (that is, a point p with $f(p) = p$). Prove: S^{n-1} is not contractible.

(d) (Not required) Find "(Bing's) house with two rooms" from literature/internet and show that it is contractible.