PROBLEM SET 10, PART 2: TOPOLOGY (H)
DUE: MAY 9 , 2022

(1) [Simply connected]
  (a) Let X be path connected. Prove that the following statements are equivalent:
    (i) X is simply connected, i.e. $\pi_1(X) = \{e\}$.
    (ii) Any loop in X can be continuously deformed to a point in X.
    (iii) For any $x_0, x_1 \in X$, any two paths $\gamma_1, \gamma_2 \in \Omega(X; x_0, x_1)$ are path homotopic.
  (b) Show that "simply connectedness" is a topological property. Is it multiplicative? preserved under continuous maps? hereditary?

(2) [The fundamental group of the product space]
  (a) Prove: $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.
  (b) (Not required) Write down a formula for the fundamental group of an arbitrary product, $\pi_1(\prod_\alpha X_\alpha, (x_\alpha))$, and prove your formula.
  [Warning: for infinitely many groups G_α, there are two ways to "multiply" them together: the direct sum $\bigoplus_\alpha G_\alpha$ and the direct product $\bigotimes_\alpha G_\alpha$.]

(3) [Base point change isomorphism]
  Let X be path connected, $x_0, x_1 \in X$. We have seen in Proposition 3.4.9 that any path λ from x_0 to x_1 induces a group isomorphism $\Gamma_\lambda : \pi_1(X, x_1) \to \pi_1(X, x_2)$.
  (a) Suppose λ_1 is a path from x_0 to x_1, and λ_2 is a path from x_1 to x_2.
  Prove: $\Gamma_{\lambda_1 * \lambda_2} = \Gamma_{\lambda_2} \circ \Gamma_{\lambda_1}$.
  (b) Prove: $\pi_1(X, x_0)$ is abelian <u>if and only if</u> for any two paths λ_1, λ_2 from x_0 to x_1, we have $\Gamma_{\lambda_1} = \Gamma_{\lambda_2}$.
  (c) Suppose X, Y are path connected, and $f \in \mathcal{C}(X, Y)$. I have a vague idea that the group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is independent of the choice of x_0. Please write down an explicit formula/rigorous statement and prove it.

(4) [The fundamental group of topological groups]
  Let G be a topological group. We want to prove $\pi_1(G, e)$ is an abelian group. There is a one-sentence proof:

  > Topological groups are group objects in the category $\mathcal{TOP}$, so under the functor π_1 (which preserves products), they become group objects in the category $\mathcal{GROUP}$, which are abelian groups.

  Unfortunately, I don't understand that fancy proof. So I want more elementary proofs. In what follows we give two proofs.
  We let γ_1, γ_2 be two loops in G based at e.

(a) (First proof) Denote by γ_e the constant loop at e. Check:

$$F(s,t) = (\gamma_1 * \gamma_e)(\max(0, t - \frac{s}{2})) \bullet (\gamma_e * \gamma_2)(\min(1, t + \frac{s}{2}))$$

is a path homotopy between $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$, where $\bullet$ is the group multiplication.

(b) (Not required)(Second proof) Construct explicit path homotopies to verify
  (i) $\gamma_1(t) \bullet \gamma_2(t) \sim \gamma_2(t) \bullet \gamma_1(t)$;
  (ii) $(\gamma_1 * \gamma_2)(t) \sim \gamma_1(t) \bullet \gamma_2(t)$.
  (Hint: $\gamma_1 * \gamma_2 = (\gamma_1 * \gamma_e) \bullet (\gamma_e * \gamma_2)$)