PROBLEM SET 11, PART 1: TOPOLOGY (H)
DUE: MAY 16, 2022

(1) [More fundamental groups]
Find the fundamental groups of the following spaces:
(a) $\mathbb{R}^{n+k} \setminus (\mathbb{R}^n \times \{(0, \cdots, 0)\})$ $(k \geq 2)$
(b) $\mathbb{R}^3 \setminus \mathbb{Z}^3$
(c) $S^2 \vee S^2$ (See section 1.4 for the definition of the wedge product)
(d) $S^1 \vee S^2$ [Hint: use the method of the proof of Prop 3.5.1]
(e) $\{(x, y, 0) \mid x, y \in \mathbb{R}\} \cup \{(0, y, z) \mid y^2 + z^2 = 1, z \geq 0\}$
(f) $\mathbb{R}^3 \setminus (\{(0, 0, z) \mid z \in \mathbb{R}\} \cup \{(x, y, 0) \mid x^2 + y^2 = 1\})$
(g) (Not required) $\mathbb{R}^3 \setminus \{(x, y, 0) | x^2 + y^2 = 1\}$
(h) (Not required) $\mathbb{R}^3 \setminus (\{(0, 0, 0)\} \cup \{(1, 1, z) \mid z \in \mathbb{R}\})$

(2) [Maps with trivial induced homomorphism]
(a) Suppose $h : S^1 \to X$ is a continuous map. Prove: The following are equivalent
  (i) The induced homomorphism $h_* : \pi_1(S^1, 1) \to \pi_1(X, h(1))$ is the trivial homomorphism (i.e. $h_*([\gamma]_p) = e$ holds for all $[\gamma]_p \in \pi_1(S^1, 1)$).
  (ii) h is null homotopic.
  (iii) h can be extended to a smooth map $H : \overline{D} \to S^1$.
(b) Now suppose $X = S^1$. Prove: (i)-(iii) are equivalent to
  (iv) h can be lifted to a continuous map $\tilde{h} : S^1 \to \mathbb{R}$ so that $p \circ \tilde{h} = h$.
(c) Read the proof of Borsuk-Ulam theorem (in which (i)$\Longrightarrow$(iv) is used) and the proof of pancake theorem on page 220-221.

(3) [The degree for maps between the circle]
For any continuous map $f : S^1 \to S^1$, there exists $n \in \mathbb{Z}$ such that $f_*([\gamma_1]_p) = [\gamma_n]_p$. The integer n is called the *degree* of the map f, and is denoted by $\deg(f)$.
(a) Prove: If $f \in \mathcal{C}(S^1, S^1)$ is not surjective, then $\deg(f) = 0$.
(b) Prove: If $f, g \in \mathcal{C}(S^1, S^1)$, then $\deg(f \circ g) = \deg(f)\deg(g)$.
(c) Prove: f is homotopic to g if and only if $\deg(f) = \deg(g)$.
(d) Read the following paragraph which gives a descriptive definition of the winding number:

> Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a continuous map and $p \notin \mathrm{Im}(\gamma)$. The *winding number* $W(\gamma, p)$ of the closed curve γ around the point p is defined to be the integer representing the total number of times that curve travels counterclockwise around the point.

Use the language of mapping degree to give a rigorous definition of winding number $W(\gamma, p)$.

(4) (Not required) [Not-so-fundamental group]
Let X be a path connected topological space, and $x_0 \in X$ be a base point. Given any two loops γ_0, γ_1 based at x_0, we define a *pseudo-homotopy* between γ_0 and γ_1 to be a map [NOT NECESSARY CONTINUOUS] $F : [0,1] \times [0,1] \to X$ s.t.

- For any fixed t, the map $\gamma_t(s) := F(t,s)$ is continuous in s.
- For any fixed s, the map $\lambda_s(t) := F(t,s)$ is continuous in t.
- For any s, $F(0,s) = \gamma_0(s),\ F(1,s) = \gamma_1(s)$.
- For any t, $F(t,0) = F(t,1) = x_0$.

We define the "*NOT-SO-Fundamental group*" of X at x_0 to be the pseudo-homotopy classes.

(a) Show that the "*NOT-SO-Fundamental group*" of S^1 is the trivial group $\{e\}$.
(b) Show that the "*NOT-SO-Fundamental group*" is not so interesting, since it is always the trivial group $\{e\}$.
(c) In proving $\pi_1(S^1) \simeq \mathbb{Z}$, where did we use the continuity of the homotopy?