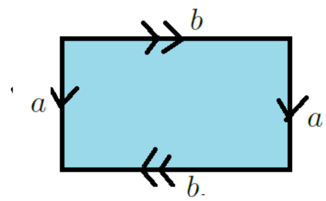
PROBLEM SET 11, PART 2: TOPOLOGY (H)
DUE: MAY 16, 2022

(1) (NOT required). [Abelianization]
Let G be a group.
  (a) Let $[G, G]$ be the subgroup of G that is generated by all elements of the form $xyx^{-1}y^{-1}$ for all $x, y \in G$. Prove: $[G, G]$ is a normal subgroup of G.
  (b) Prove: The group $Ab(G) := G/[G, G]$ is abelian (called the *abelianization* of G).
  (c) Prove: The abelianization defines a functor from $\mathcal{GROUP}$ to $\mathcal{ABELGROUP}$.
  (d) What is the abelianization of $\mathbb{Z} * \cdots * \mathbb{Z}$?
  (e) Prove: $Ab(\langle a_1, b_1, \cdots, a_n, b_n | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1\rangle) = \mathbb{Z}^{2n}$.
  (f) Prove: $Ab(\langle a_1, \cdots, a_n | a_1^2 \cdots a_n^2 = 1\rangle) = \mathbb{Z}^{n-1} \times \mathbb{Z}_2$.

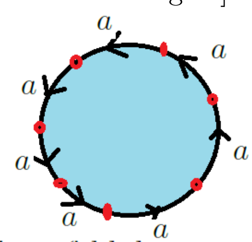
(2) [The wedge sum of circles]
  (a) Finite wedge sum and applications.
    (i) Prove: $\pi_1(S^1 \vee S^1 \vee \cdots \vee S^1) \simeq \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$.
    (ii) What is the fundamental group of $\mathbb{R}^2 - \{\text{finitely many points}\}$?
    (iii) (NOT required) What is the fundamental group of $\mathbb{R}^2 - \mathbb{Z}^2$?
    (iv) (Not required) What is the fundamental group of the set $\mathbb{R}^3 - \{\text{finitely many lines passing } 0\}$?
    (v) (NOT required). A group is called *finitely presented* if it has a presentation $G = \langle S | R\rangle$ where both S and R are finite sets. Prove: any finitely presented group is the fundamental group of some compact Hausdorff space.
    [Hint: First construct a wedge sum of circles with fundamental group $\langle S\rangle$, then for each element in R attach a disk to kill the relation.]
  (b) Infinite wedge sum.
    (i) Let $X = \cup_{n\geq 1} C_n$, where C_n is the circle in $\mathbb{R}^2$ of radius n centered at $(n, 0)$. Compute $\pi_1(X)$.
    (ii) Let $Y = \{(x, 0) \mid x \in \mathbb{R}\} \cup \bigcup_{n\geq 1} \widetilde{C}_n$, where $\widetilde{C}_n$ is the circle in $\mathbb{R}^2$ of radius $1/3$ centered at $(n, 1/3)$. Compute $\pi_1(Y)$. Are X and Y homeomorphic? homotopic equivalent?
    (iii) (NOT required). Let $Z = \cup_{n\geq 1} C_{1/n}$, where $C_{1/n}$ is the circle of radius $1/n$ centered at $(1/n, 0)$. Prove: There is a surjective homeomorphism from $\pi_1(Z)$ to the *direct product* $\prod_{n\geq 1} \mathbb{Z}$. As a consequence, $\pi_1(Z)$ contains uncountably many elements [So Z is not homotopy equivalent to X or Y].
    (iv) (NOT required).Use (iii) to prove: $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ contains uncountably many elements.

(3) [Application of van Kampen]
Use van Kampen theorem to compute the fundamental group of

(a) $\mathbb{RP}^2$
(b) The Klein bottle.
(c) (Not required) The n-fold dunce cap. [Split the boundary circle of a closed disk into n parts (by n red dots), and identify the boundary segments according to the picture below (but keep the interior of the disk unchanged.]



The Klein bottle



The n-fold dunce cap

(d) Prove: The fundamental group of $\Sigma_g = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$ is given by

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$

(e) Remove k small disjoint discs from Σ_g and denote the resulting space by $\Sigma_{g,m}$. Compute $\pi_1(\Sigma_{g,m})$
(f) (Not required) Compute the fundamental group of $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$

(4) [The fundamental group of topological manifolds]
Let X, Y be connected topological manifolds.
(a) Suppose $\dim X > 2$. Prove: For any point $x \in X$, $\pi_1(X) \simeq \pi_1(X \setminus x)$.
(b) Prove: $\pi_1(X \vee Y) \simeq \pi_1(X) * \pi_1(Y)$.
(c) Suppose $\dim X = \dim Y > 2$. Prove: $\pi_1(X \# Y) \simeq \pi_1(X) * \pi_1(Y)$.
(d) (NOT required) Prove: The fundamental group of any topological manifold is countable (i.e. contains only countably many elements).
[Hint: cover X by countably many open sets U_i that are homeomorphic to Euclidean balls. Pick a point from each U_i and from each component of all possible $U_i \cap U_j$. Try to show that each loop is path homotopic to loops consisting of segments connecting the chosen points.]