PROBLEM SET 12, PART 1: TOPOLOGY (H)
DUE: MAY 23, 2022

(1) [Products of coverings]
   (a) Prove: If X is connected, $\widetilde{X} \neq \emptyset$, then p is surjective, and the cardinality of $p^{-1}(x)$ is independent of x.
   (b) Prove: If $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X'$ are covering maps, so is their product $p \times p' : \widetilde{X} \times \widetilde{X}' \to X \times X'$.
   (c) Let $p : \mathbb{R} \to S^1$ be the standard covering map. Prove: The infinite product $\prod_{n\in\mathbb{N}} p : \prod_{n\in\mathbb{N}} \mathbb{R} \to \prod_{n\in\mathbb{N}} S^1$ is NOT a covering map.

(2) [Fundamental groups of covering spaces]
Suppose $X, \widetilde{X}$ are path-connected, $p : \widetilde{X} \to X$ is a covering map, and $p(\tilde{x}_0) = x_0$.
   (a) Suppose γ is a loop in X based at x_0. Prove: γ can be lifted to a *loop* in $\widetilde{X}$ based at $\tilde{x}_0$ if and only if $[\gamma] \in p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.
   (b) Prove: the index of the subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ is the cardinality of $p^{-1}(x_0)$.
   (c) Prove: If the base space X is simply connected, then p is a homeomorphism.
   (d) Suppose $\tilde{x}_1 \in p^{-1}(x_0)$. Prove: as subgroups of $\pi_1(X, x_0)$, the two groups $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ and $p_*(\pi_1(\widetilde{X}, \tilde{x}_1))$ are conjugate to each other.

(3) [properly discontinuous actions]
   (a) Let $G = \langle a, b | a^{-1}bab = 1\rangle$. Consider the action of G on $\mathbb{R}^2$ generated by

$$a \cdot (x, y) := (-x, y - 1), \qquad b \cdot (x, y) = (x + 1, y).$$

      (i) Show that this action is properly discontinuous, and the quotient space is the Klein bottle. What is the fundamental group of the Klein bottle?
      (ii) Also check that the quotient space in Example 3.7.6 is the Klein bottle, and thus $\mathbb{T}^2$ is a double covering of the Klein bottle.
   (b) (Not required) Suppose group G acts on $\widetilde{X}$. We say the action is *free* if

$$\text{for any } g \neq e \text{ and any } x \in \widetilde{X},\ g \cdot x \neq x.$$

   Prove: If $\widetilde{X}$ is Hausdorff, G is a finite group, and the G-action on $\widetilde{X}$ is free, then the action is properly discontinuous.
   (c) (Not required) More generally, Let $\widetilde{X}$ be a LCH space. Suppose the G-action on $\widetilde{X}$ is free, and satisfies the following condition (known as *proper action*):

$$\text{for any compact subset } C \subset \widetilde{X}, \text{ the set } \{g \mid g \cdot C \cap C \neq \emptyset\} \text{ is finite,}$$

   Prove: the G-action is properly discontinuous, and $\widetilde{X}/G$ is a LCH space.
   [Hint: By locally finiteness, for any compact C, $\cup_g g \cdot C$ is closed.]

(4) [$SU(2)$ and $SO(3)$](Not required)

Let $SU(2)$ be the special unitary group, i.e. the group of 2×2 unitary matrices with determinant 1, and $SO(3)$ the special orthogonal group, i.e. the group of 3×3 orthogonal matrices with determinant 1.

(a) Prove: $SU(2)$ is homeomorphic to S^3 (and thus is simply connected).

(b) Prove: $SU(2)$ is a double covering of $SO(3)$ (and thus $SO(3) \simeq \mathbb{RP}^3$).

(c) What is the fundamental group of $SO(3)$?