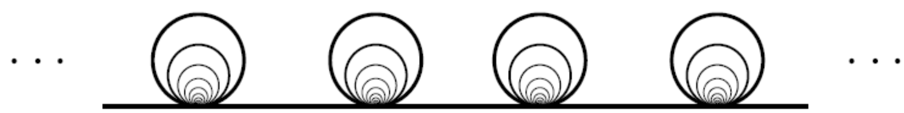
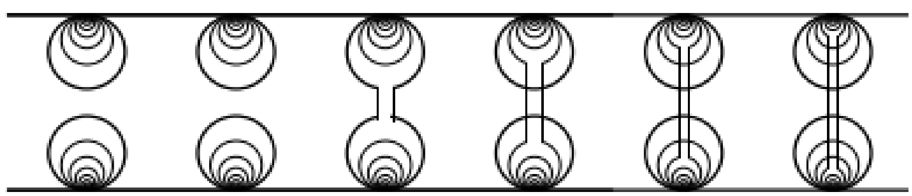
PROBLEM SET 12, PART 2: TOPOLOGY (H)
DUE: MAY 23, 2022

(1) [Covering of covering space]
Let X, Y, Z be path-connected and locally path-connected spaces, and $f : X \to Y$, $g : Y \to Z$ be continuous maps.
  (a) Suppose both g and $g \circ f$ are covering maps. Prove: f is a covering map.
  (b) Suppose both f and $g \circ f$ are covering maps. Prove: g is a covering map.
  (c) Suppose f is a covering, and g is finite covering. Prove: $g \circ f$ is a covering.
  (d) Suppose f and g are covering, and suppose Z is semi-locally simply connected. Prove: $g \circ f$ is a covering.
  (e) (Not required) Let X be the second space below, Y be the first space below, and Z be the Hawaii earing. Construct a natural covering map $g : Y \to Z$, and a natural double covering map $f : X \to Y$ (as a double covering), so that the composition $g \circ f$ is NOT a covering map. [So in general the composition of covering maps may fail to be a covering map.]





(2) [Classify covering spaces]
  (a) Find all path connected covering spaces of $S^1 \vee S^2$.
  (b) Find all path connected covering spaces of $\mathbb{T}^2 = S^1 \times S^1$.
  [You may use the fact that the subgroups of $\mathbb{Z}^2$ are
  - $\{(0,0)\}$,
  - $\{k(p,q) \mid k \in \mathbb{Z}\}$ (where $(p,q) \in \mathbb{Z}^2$)
  - $\{k_1(p,q) + k_2(r,s) \mid k_1, k_2 \in \mathbb{Z}\}$ (where $(p,q),(r,s) \in \mathbb{Z}$, and $ps - qr \neq 0$.]

(3) [Covering of topological groups and manifolds]

(a) (Not required) Let G be a topological group which is path-connected and locally path-connected.

(i) Suppose $\widetilde{G}$ is path-connected, and let $p : \widetilde{G} \to G$ be a covering map. Fix an element $\tilde{e} \in p^{-1}(e)$. Define a map $m : \widetilde{G} \times \widetilde{G} \to G$ by

$$m(\tilde{a}, \tilde{b}) := p(\tilde{a}) \cdot p(\tilde{b})$$

Prove: m can be lifted to a map $\tilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ with $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$.

(ii) Prove: Any covering space of a topological group is a topological group.

(b) Let M be a topological manifold.

(i) Prove: Any topological manifold admits a universal covering.

(ii) Prove: Any covering space of a topological manifold is still a topological manifold. [It follows that any Lie group admits a universal covering which is still a Lie group. This fact plays an important role in classifying Lie groups.][Hint:what do we know about the fundamental group of a topological manifold?]

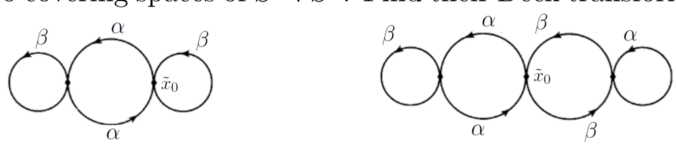
(4) [Deck transformation] (Not required)

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Its *Deck transformation group* is

$$\mathrm{Aut}(p) := \{h : \widetilde{X} \to \widetilde{X} \mid h \text{ is a covering space isomorphism}\}.$$

(a) For each path connected covering space of S^1, find its Deck transformation group.

(b) Below are two covering spaces of $S^1 \vee S^1$. Find their Deck transformation groups.



(c) Suppose G acts on $\widetilde{X}$ which is path-connected, and suppose the action is properly discontinuous. Prove: G is the deck transformation group of the covering $p : \widetilde{X} \to X = \widetilde{X}/G$.

(d) Let $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal covering. Define an action of $G = \pi_1(X, x_0)$ on $\widetilde{X}$, and prove that the action you defined is properly discontinuous..

[Thus the deck transformation group of the universal covering is $\pi_1(X, x_0)$.]