PROBLEM SET 13, PART 1: TOPOLOGY (H)
DUE: MAY 30, 2022

(1) [Applications of Brouwer's Fixed Point Theorem]
   (a) [A special case of Poincare-Hopf Theorem, proved by Hadamard] Let $f : \overline{B^n} \to \mathbb{R}^n$ be a continuous map (i.e. f is a vector field on $\overline{B^n}$) such that $x \cdot f(x) > 0$ for all $x \in S^{n-1} = \partial \overline{B^n}$. Prove: there exists $x \in B^n$ such that $f(x) = 0$.
   (b) [Poincare-Bohl] Let $f : \overline{B^n} \to \mathbb{R}^n$ be a continuous map such that $f(x) \notin \{\alpha x \mid \alpha > 0\}$ for any $x \in S^{n-1}$. Prove: there exists $x \in \overline{B^n}$ such that $f(x) = 0$.
   (c) [Perron-Frobenius] Any $n \times n$ real matrix with positive entries has a positive eigenvalue, and the corresponding eigenvector can be chosen to have strictly positive entries.
   (d) [Kuratowski-Steinhuas] Let $f : \overline{B^n} \to \overline{B^n}$ be a continuous map such that $f(S^{n-1}) \subset S^{n-1}$, and suppose for any $x \in S^{n-1}$, $f(x) \neq x$. Prove: $f(\overline{B^n}) = \overline{B^n}$.

(2) [Brouwer's Fixed Point Theorem, 2^{nd} version](Not required)
Let $K \subset \mathbb{R}^n$ be any non-empty compact convex subset.
   (a) Suppose K has non-empty interior. Prove: K is homeomorphic to $\overline{B^n}$.
   (b) Prove: K has non-empty interior if and only if K is not contained in a proper hyperplane (i.e. a set of the form $x_0 + V$, where $V \subset \mathbb{R}^n$ is a linear subspace).
   (c) Prove Theorem 4.1.5.

(3) [Poincaré-Miranda theorem]
The following theorem was first announced by H. Poincaré in 1883, which can be viewed at first glance as a higher dimension generalization of intermediate value theoremz. Miranda showed in 1940 that the theorem was equivalent to the Brouwer's fixed point theorem.

Poincaré-Miranda Theorem. Let $f = (f_1, \cdots, f_n) : [0, 1]^n \to \mathbb{R}^n$ be continuous. Suppose for any $1 \le i \le n$, we have

$$f_i \le 0 \quad \text{on} \quad \{x \in [0,1]^n \mid x_i = 0\},$$
$$f_i \ge 0 \quad \text{on} \quad \{x \in [0,1]^n \mid x_i = 1\}.$$

Then there exists $p \in [0, 1]^n$ such that $f(p) = 0$.

   (a) Prove Poincaré-Miranda theorem via Brouwer's fixed point theorem.
   [Hint: Let $r : \mathbb{R} \to [0, 1]$ be the retraction with $r((-\infty, 0)) = 0, r((1, +\infty)) = 1$ and let $r(x) = (r(x_1), \cdots, r(x_n))$. Consider $h(x) = r(x) - f(r(x))$. Then h maps into a large ball into itself. Show that the fixed point of h lies in $[0, 1]^n$.]
   (b) Prove Brouwer's fixed point theorem via Poincaré-Miranda theorem.

(4) [Applications of Brouwer's invariance of domain theorem]
 (a) Prove: there is no injective continuous map $f : S^n \to \mathbb{R}^n$.
  - Then show that there is no proper subset of S^n that is homeomorphic to S^n itself.

 (b) Show that conception of the boundary point is well-defined in the definition of "topological manifold with boundary".
  - Then show that if X is a topological manifold with boundary of dimension n, then its boundary ∂X is a topological manifold of dimension $n-1$.