PROBLEM SET 13, PART 2: TOPOLOGY (H)
DUE: MAY 30, 2022

(1) [A story about love and hates]
In a certain country there are two towns, A and B, and two disjoint roads, α and β, connecting them. Two lovers in town A must travel to town B, one by road α and one by road β. So great is the force of their love that if at any instant they are separated by ten kilometers or more, they will surely die. There are also two enemies, one lives in town A and must travel to town B by road α, while the other lives in town B and must travel to town A by road β. So great is the force of their hatred that if at any instant they are separated by ten kilometers or less, they will surely die.

(a) Show that at least two people will end up dead by converting the previous problem to the following one:

> Let $\gamma_1 : [0,1] \to [0,1]^2$ be a path from the point $(0,0)$ to the point $(1,1)$, and $\gamma_2 : [0,1] \to [0,1]^2$ be a path from the point $(0,1)$ to the point $(1,0)$.
>
> **Claim:** γ_1 and γ_2 must intersect.

(b) Here is a fake proof the claim above:

> Since γ_1 is a path in the square $[0,1]^2$ and since paths are continuous, we may find a continuous function $f : [0,1] \to [0,1]$ so that the image of the path γ_1 is the graph of f. Similarly we may find a continuous function g whose graph is the path γ_2. By assumption, we have $f(0) = 0$, $f(1) = 1$ and $g(0) = 1$, $g(1) = 0$. Consider the function $h(x) := f(x) - g(x)$. Then h is a continuous function with $h(0) = -1$, $h(1) = 1$, so there is $x_0 \in [0,1]$ so that $h(x_0) = 0$, i.e. $f(x_0) = g(x_0)$. So the paths γ_1 and γ_2 intersect at the point $(x_0, f(x_0))$.

Find the <u>mistake</u> in this proof.

(2) [Brouwer's Invariance of domain theorem revisited]

(a) (Higher dimensional analogue of "arc non-separation" theorem) Prove: If $K \subset \mathbb{R}^n$ is compact and is a retract of $\mathbb{R}^n$, then $\mathbb{R}^n \setminus K$ is connected.

(b) Let $D = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ be the open unit disc. Use Jordan curve theorem to prove: If $f : \overline{D} \to \mathbb{R}^2$ is continuous and injective, then $f(D)$ is the interior (=the bounded component) of the Jordan curve $f(S^1)$. [Hint: $f(\overline{D})$ is a retract of $\mathbb{R}^2$.]

(c) (Not required) Assume Jordan-Brouwer Theorem holds. State a higher dimensional analogue of (b) and prove it.

(3) [Application to the square peg problem]
Let $J \subset \mathbb{R}^2$ be a Jordan curve that is symmetric about the origin (i.e. $P \in J$ if and only if $-P \in J$). Moreover, assume the origin O lies in the bounded connected component of $\mathbb{R}^2 \setminus J$. Prove: J has an inscribed square, i.e. there exists four points

on J that are the vertices of a square.
[Hint: rotate the curve C by $\pi/2$ and try to find an intersection point.]

(4) [Applications to graph theory](Not required)
We say a graph $G = (V, E)$ is a *planar graph*, if it can be embedded into $\mathbb{R}^2$, i.e. can be drawn in $\mathbb{R}^2$ so that no edge cross.
  (a) Prove: The graph K_5 (=the graph with vertices $V = \{a_i \mid 1 \le i \le 5\}$ and edges $\{a_i a_j \mid 1 \le i < j \le 5\}$) is not a planar graph.
  (b) We say a space $X \subset \mathbb{R}^2$ is a θ-*space* if X is the union of three arcs A, B, C, so that they intersect and only intersect each other at their end points (so that the space looks like the letter "θ"). Prove: If $X \subset \mathbb{R}^2$ is a θ-space with arcs A, B, C, then $\mathbb{R}^2 \setminus X$ has three connected components, whose boundaries are $A \cup B$, $B \cup C$ and $C \cup A$ respectively.
  (c) Prove: The graph $K_{3,3}$ (=the graph with vertices $V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and edges $\{a_i b_j \mid 1 \le i, j \le 3\}$) is not a planar graph.