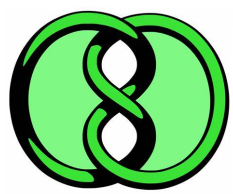
PROBLEM SET 14, PART 3: TOPOLOGY (H)
DUE: JUNE 8, 2022

(1) [Symbolic presentation of surfaces]
Find out the surfaces in our list (i.e. $S^2, \Sigma_k, \widetilde{\Sigma}_l, S^2_m, \Sigma_{k,m}, \widetilde{\Sigma}_{l,m}$ with specified k, l, m) which are homeomorphic to the ones given below:
- (a) $\langle a, b, c, d \mid acadbcb^{-1}d\rangle$.
- (b) $\langle a, b, c, d, e \mid abcb^{-1}adede^{-1}\rangle$.
- (c) See picture below.
- (d) See picture below.


(c)


(d)

(2) [Orientability]
- (a) Prove Proposition 4.4.21.
- (b) (Not required) For each orientable compact surface (without boundary) $\Sigma_{k,m}$, prove: there exists an orientation-reversing self-homomorphism (that is, a homeomorphism $f : \Sigma_{k,m} \to \Sigma_{k,m}$ so that for some oriented triangulation K of σ, f maps simplexes of K to simplexes of K, such that for each triangle ABC in K, the orientation on $\langle f(A), f(B), f(C)\rangle$ is $-[f(A)f(B)f(C)]$)
[Hint: Just put the surface at a nice position, and consider the map that maps one coordinate to its inverse. You need to handle even/odd number of boundary circles separately.]

(3) [Euler characteristic v.s. covering]
We know that S^2 is a double covering of $\mathbb{RP}^2$. In Section 3.7 we have seen that Σ_{11} is a 5-fold covering space of Σ_3.
- (a) Compare $\chi(S^2)$ and $\chi(\mathbb{RP}^2)$. Compare $\chi(\Sigma_{11})$ and $\chi(\Sigma_3)$.
- (b) In general, suppose S_1, S_2 are compact connected surfaces without boundary, and $p : S_1 \to S_2$ is a k-fold covering. Find the relations between $\chi(S_1)$ and $\chi(S_2)$, and prove it.
- (c) In general, if Σ_m is a covering space of Σ_n, Find the relation between m and n.
- (d) [Not required] For each non-orientable connected surface without boundary, i.e. $\widetilde{\Sigma}_{l,m}$, there exists an orientable connected surface which is a double covering of $\widetilde{\Sigma}_{l,m}$. Which surface is it?

(4) [Triangulation of surface] [Not required]
Let K be a triangulation of a compact surface S without boundary, and let $|V|, |E|, |F|$ be the number of vertices, edges and triangles in K. Prove:
  (a) $3|F| = 2|E|$.
  (b) $|E| = 3(|V| - \chi(S))$.
  (c) $|V| \geq \frac{7+\sqrt{49-24\chi(S)}}{2}$. [So we have seen the triangulation of $\mathbb{T}^2$ and $\mathbb{RP}^2$ with least vertices.][This is also related to the following question: how many color do you need to color a map on surface S?]

[**Last Problem**]
We learned many beautiful theorems in this course. Write down at least two of them, one from part one of this course, and the other from part two of this course.