(1) **Measure zero set in smooth manifolds**
   (a) Prove: the phrase “measure zero” is well-defined on smooth manifolds.
   (b) Deduce Sard’s theorem from the Euclidean case.
   (c) Show that if \( f : M \to N \) is a smooth map of constant rank \( r < \dim N \), then \( f(M) \) has measure zero.

(2) **An counterexample to Sard’s Theorem**
Here is a counterexample to Sard’s theorem if \( f \) is not smooth enough (constructed by E. Grinberg). Let \( C \subset [0,1] \) be the Cantor set.
   (a) Construct a \( C^1 \) function \( f : \mathbb{R} \to \mathbb{R} \) such that the critical set of \( f \) contains \( C \).
       (Hint: Denote \([0,1] \setminus C = \cup_{k=1}^{\infty} (a_k, b_k)\). Start with a “continuous bump function” \( f_k \) on \((a_k, b_k)\) with \( \int f_k(t) \, dt = b_k - a_k \).
   (b) Show that the function \( g : \mathbb{R}^2 \to \mathbb{R} \) defined by \( g(x, y) = f(x) + f(y) \) is \( C^1 \), and the set of critical values contains an interval. (Hint: Show that \( C + C = [0,2] \).)

(3) **Morse functions**
Let \( U \subset \mathbb{R}^n \) be an open set, and \( f \in C^\infty(U) \).
   - We say a critical point \( p \in U \) of \( f \) is non-degenerate if the Hessian matrix
     \[
     \text{Hess}_f(p) = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right)(p)
     \]
     is non-degenerate.
   - A function is called a Morse function if every critical point is non-degenerate.

Prove:
   (a) Use inverse function theorem to prove that non-degenerate critical point must be isolated.
   (b) Given any \( f \in C^\infty(U) \), for almost every \( a \in \mathbb{R}^n \), the “linear perturbation”
     \[
     f_a : U \to \mathbb{R}, \quad x \mapsto f(x) + a_1 x^1 + \cdots + a_n x^n
     \]
     of \( f \) is a Morse function on \( U \).
       (Hint: Consider regular values of the map \( g = df = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}) : U \to \mathbb{R}^n \).)
   (c) (Not required) Suppose \( U \) is bounded. Prove: for any \( f \in C^\infty(U) \) and any \( \varepsilon > 0 \), there is a Morse function \( g \in C^\infty(U) \) so that \( |g - f| < \varepsilon \) and all critical values of \( g \) are distinct.
   (d) (Not required) Extend the result in (c) to smooth functions defined on a compact manifold.
(4) **The Lagrange multiplier**

Let $M$ be a smooth manifold, and $f \in C^\infty(M)$ a smooth function. We would like to study the critical points of the function $\tilde{f} := f|_S \in C^\infty(S)$ for a smooth submanifold $S \subset M$. For simplicity, we suppose there is a smooth map $g : M \to N$ and a regular value $p \in N$ of $g$ so that $S = g^{-1}(q)$. Prove: a point $p \in S$ is a critical point of $\tilde{f}$ if there exists a linear function $L : T_qN \to \mathbb{R}$ (called a Lagrange multiplier), so that $df_p = L \circ dg_p$.

(5) **Proper maps**

Recall that a map is called *proper* if the pre-image of any compact set is compact. Let $f : M \to N$ be a smooth and proper map.

(a) Prove: If an injective immersion $f : M \to N$ is proper, then it is an embedding.

(b) Now suppose $\dim M = \dim N$, and suppose $q \in f(M)$ be a regular value of $f$. Prove: $f^{-1}(q)$ is a finite set $\{p_1, \ldots, p_k\}$, and there exist a neighborhood $V$ of $q$ in $N$ and neighborhoods $U_i$ of $p_i$ in $M$ such that

- $U_1, \ldots, U_k$ are disjoint coordinate charts in $M$,
- $f^{-1}(V) = U_1 \cup \cdots \cup U_k$,
- For each $1 \leq i \leq k$, $f$ is a diffeomorphism from $U_i$ onto $V$.

(6) **The cotangent bundle**

Let $M$ be a smooth manifold of dimension $n$. Let $T^*_pM$ be the dual vector space of $T_pM$, with a dual basis $\{dx^1, \ldots, dx^n\}$ (which is defined locally for a coordinate chart of $M$) which is defined to be the dual of $\{\partial_1, \ldots, \partial_n\}$. Let $T^*M = \bigcup_T T^*_pM$ be the disjoint union of all $T^*_pM$. We will call $T^*M$ the *cotangent bundle* of $M$.

(a) Modify PSet2-1-3 to endow with $T^*M$ a topology so that it is a smooth manifold of dimension $2n$.

(b) Prove: $T^*M$ is orientable.

(c) **(Not required)** Prove: If $f$ is a smooth function on $M$, then the map

$$ s_f : M \to T^*M, \quad p \mapsto (p, df_p) $$

is an injective immersion and is proper. [In particular, its image is a smooth submanifold of $T^*M$.]

(d) **(Not required)** For any $(p, \xi_p) \in T^*M$, the tangent space $T_{(p, \xi_p)}T^*M \cong T_pM \oplus T^*_pM$. 