

PROBLEM SET 8, PART 2: TOPOLOGY (H)
DUE: NOVEMBER 8, 2024

Do as much as you can. Problems marked (*) are not required.

(1) [Productive properties]

A topological property P is called *productive* if

Each $(X_\alpha, \mathcal{T}_\alpha)$ satisfies $P \implies (\prod_\alpha X_\alpha, \mathcal{T}_{product})$ satisfies P .

- (a) Prove: (T1), (T2) and (T3) are productive.
- (b) Conversely, if $(\prod_\alpha X_\alpha, \mathcal{T}_{product})$ is (T1), (T2) or (T3), can we conclude that each $(X_\alpha, \mathcal{T}_\alpha)$ is (T1), (T2) or (T3)?
- (c) Is (T4) productive? Is Lindelöf productive?
- (d) (*) Prove: *separable* and *metrizable* are not productive. What about (A1), (A2)?
- (e) (*) Can you introduce a weaker version of productivity, so that those non-productive properties in part (d) satisfy the weaker one?

(2) [Baire space]

A topological space is called a *Baire space* if every intersection of a countable collection of open dense sets in the space is also dense.

- (a) Use “open-closed” duality to give an equivalent characterization of Baire space.
- (b) Prove: Any complete metric space is a Baire space.
- (c) Prove: Any compact Hausdorff space is a Baire space.
- (d) (*) Prove: Any locally compact Hausdorff space is a Baire space.

(3) [Applications of Urysohn lemma]

- (a) Let X be a compact Hausdorff space, $x_0 \in X$, and U is an open neighborhood of x_0 . Prove: For any $\varepsilon > 0$ and any continuous function $f : X \rightarrow \mathbb{R}$, there exists a continuous function $g : X \rightarrow \mathbb{R}$ satisfying all of the following three conditions:

- $\sup_{x \in X} |g(x) - f(x)| < \varepsilon$.
- $g = f$ on U^c .
- there exists a neighborhood V of x_0 such that $g(x) \equiv f(x_0)$ on V .

- (b) (*) Let X be LCH. Recall

- $\mathcal{C}_b(X, \mathbb{R}) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous and bounded}\}$.
- $\mathcal{C}_c(X, \mathbb{R}) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous and compactly supported}\}$.
- $\mathcal{C}_0(X, \mathbb{R}) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous and vanishes at infinity}\}$.

On $\mathcal{C}_b(X, \mathbb{R})$ we have a metric $d_\infty(f, g) := \sup_{x \in X} |f(x) - g(x)|$. Prove: The closure of $\mathcal{C}_c(X, \mathbb{R})$ in $\mathcal{C}_b(X, \mathbb{R})$ is $\mathcal{C}_0(X, \mathbb{R})$.

(4) (*) Let $(X, \mathcal{T})$ be a compact Hausdorff space. Show that the following are equivalent:

- (a) X is metrizable.
- (b) The diagonal $\Delta \subset X \times X$ is a G_δ set.
- (c) There exists a continuous function $f : X \times X \rightarrow \mathbb{R}$ such that $f^{-1}(0) = \Delta$.